Identifying good directions to escape the NTK regime and efficiently learn low-degree plus sparse polynomials

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NeurIPS 2022

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arxiv.org/abs/2206.03688

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Optimization | **Generalization**

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Q: Can we encourage each neuron to move further and escape the NTK regime?
Does this allow us to break NTK sample complexity lower bounds?

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- Recall that the NTK is the linearization of network at initialization:

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Goal #1: Regularize to prevent movement in small eigenvalue directions.

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Goal #2: Move in $\mathbf{Q}_3$ directions, but minimally in $\mathbf{Q}_2$ directions.

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NTK is minimax optimal for **dense polynomials**, QuadNTK can learn **sparse polynomials**.
Question: Can we jointly use **both** terms to learn a larger class of functions?

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- NTK or QuadNTK alone require $n \geq \Omega(d^{k+1})$ samples to learn f^*.
- We obtain better sample complexity than either NTK or QuadNTK on their own $\implies$ **best of both worlds!**

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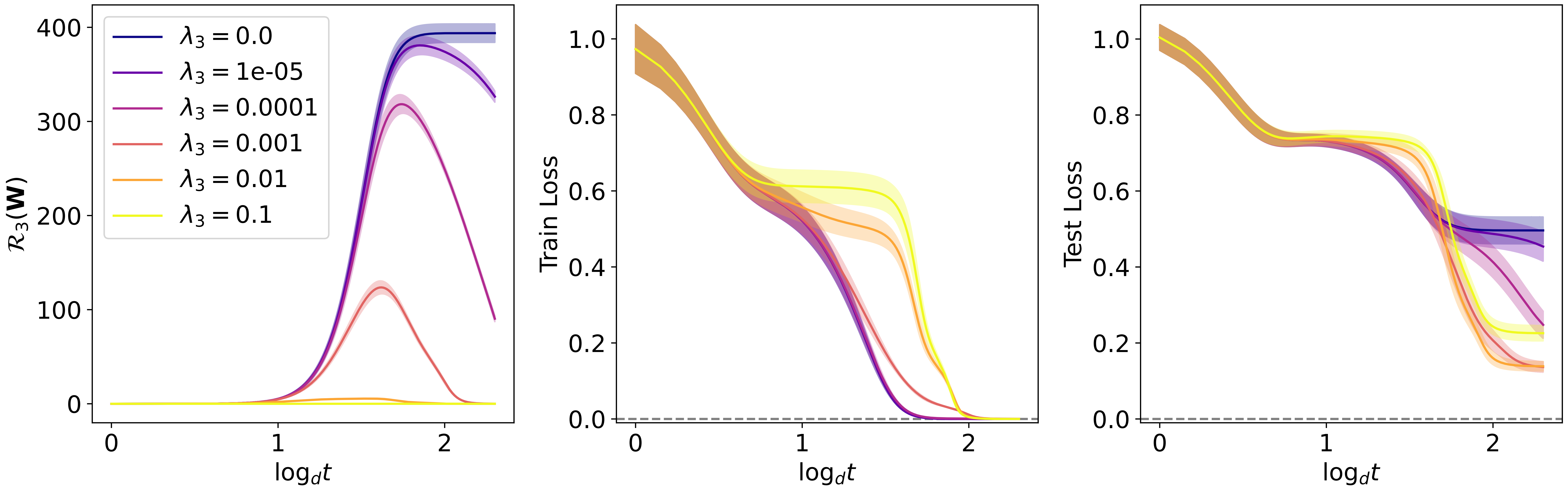
4. Generalization

Small regularized training loss implies small population loss.

Experiments

$f_L + f_Q$ trained on a degree 2 signal with $d^{1.5}$ samples



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  - How does the QuadNTK relate to feature learning?

Thanks for Listening!

References:

[1] Sanjeev Arora, Simon S. Du, Wei Hu, Zhiyuan Li, Ruslan Salakhutdinov, and Ruosong Wang. On exact computation with an infinitely wide neural net. In Advances in Neural Information Processing Systems (NeurIPS), 2019.
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