Learning and Covering Sums of Independent Random Variables with Unbounded Support

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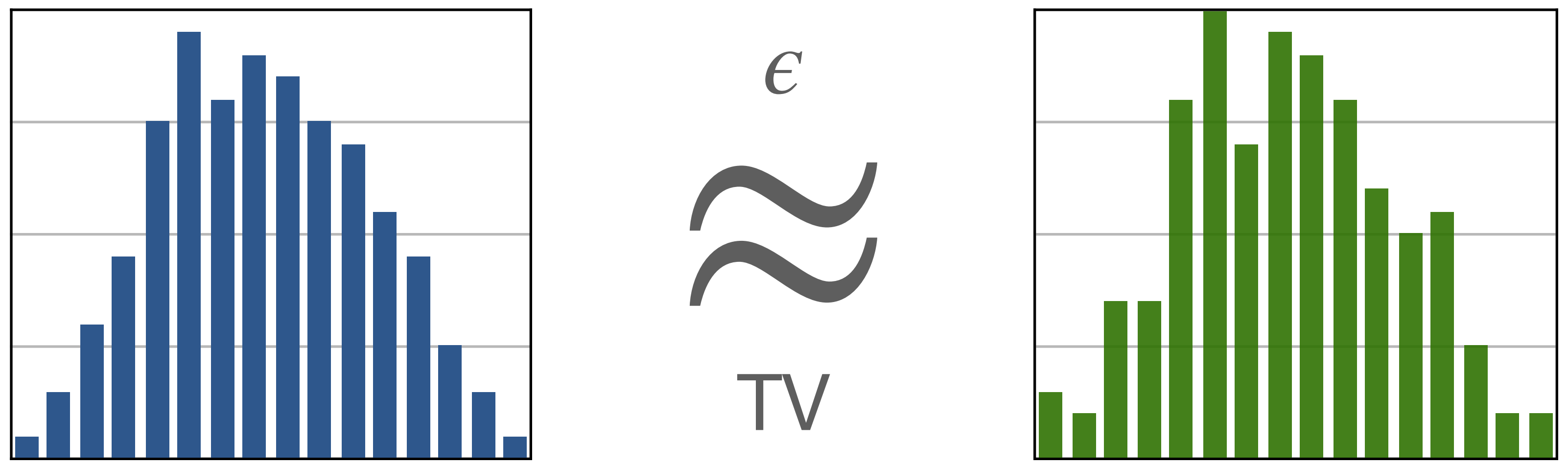
Setup

Sums of Independent Integer Random Variables (SIIRVs)

We focus on a fundamental specific type of integer random variables:

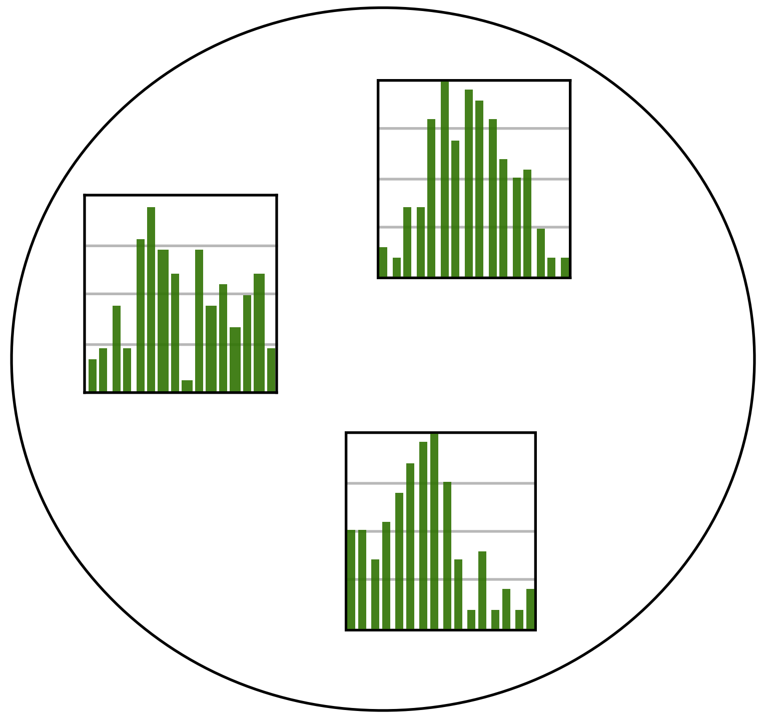
$$\sum_{i=1}^{n} X_i \text{ with independent, integer valued terms}$$

Tasks: 1. *Density estimation* 2. *Sparse Covering*



Distribution of Samples
(from the sum)

Output Distribution



Small Number of
Representatives

Setup

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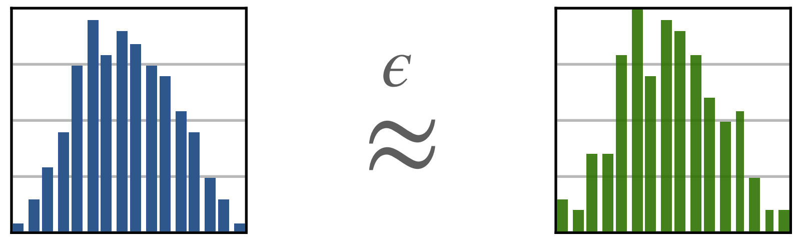
Tasks:

1. *Density estimation*

Given m i.i.d samples from $X = \sum_{i=1}^{n} X_i$, output Y such that

$$d_{TV}(Y, X) \leq \epsilon$$

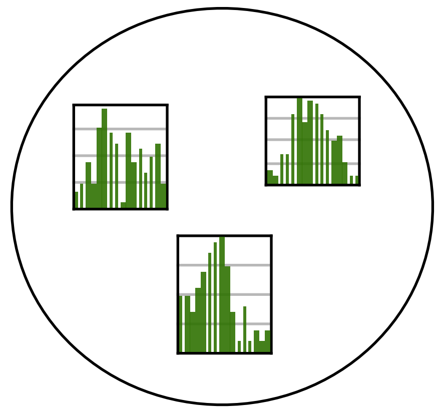


Distribution of Samples (from the sum) $\overset{\epsilon}{\approx}$ Output Distribution

2. *Sparse Covering*

For a family $\mathscr{F}$ of SIIRVs, identify a small set of distributions $\mathscr{C}$ ($|\mathscr{C}| < \infty$) so that for any $X \in \mathscr{F}$, there exists $Y \in \mathscr{C}$ with

$$d_{TV}(Y, X) \leq \epsilon$$



$\mathscr{C}$

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Tasks: 1. *Density estimation* 2. *Sparse Covering*

Challenge 1: $m = \Theta_n(1)$
Sample Complexity
independent from n

Challenge 2: $\mathscr{C} \subseteq \mathscr{F}$
Representatives are members of
the considered family of SIIRVs
(proper covering)

Motivation

Sums of Independent Integer Random Variables (SIIRVs)

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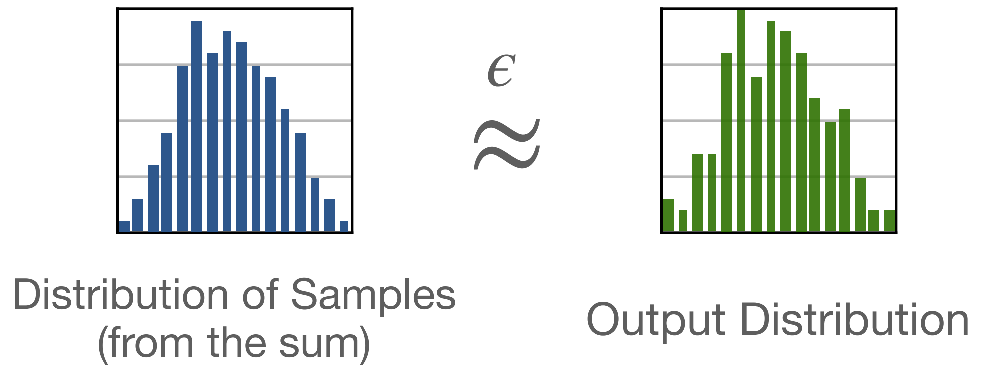
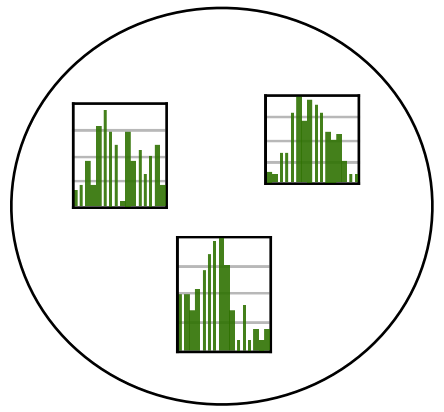
Challenge 1.
Sample Complexity
independent from n



Challenge 2.
Representatives are
SIIRVs themselves
(proper covering)



Applications (of challenges 1 *and* 2) in:

Mechanism Design:
Designing Auctions
[GT15]

Game Theory:
Computing Equilibrium in
Anonymous Games
[DDKT16],[DKS16],
[GT17],[CDS17]

Stochastic Optimization
[De18]

Previous results

Sums of Independent Integer Random Variables (SIIRVs)

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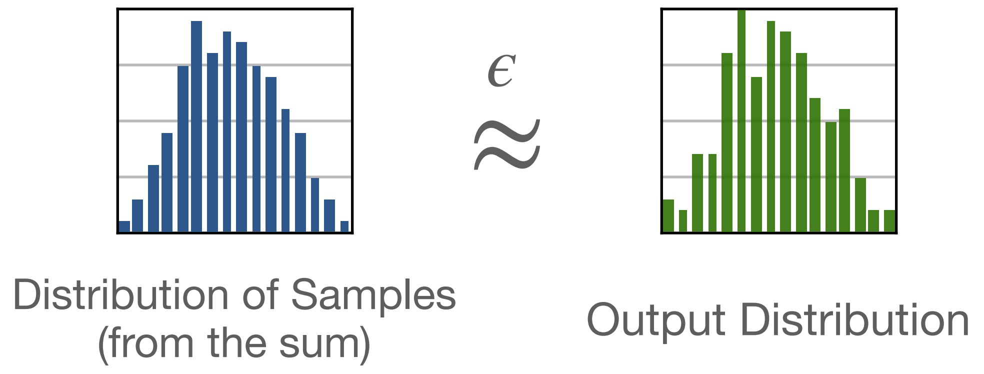

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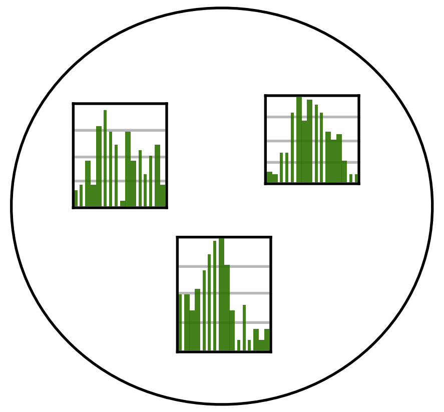

Restrict the distributions of the terms X_i to be

Bernoulli:
Learning and Covering
Poisson Binomial Distributions
[DP15, DDS15, DKS16]

poly$(1/\epsilon)$ samples & *proper* sparse covers

Supported on $\{0,\ldots,m-1\}$:
Bounded Support
[DDO+13, DKS16]

poly(m/ϵ) samples & sparse covers

High dimensional (not integer), again with bounded support:
[DKT15, DDKT16, DKS16]

Previous results

Sums of Independent Integer Random Variables (SIIRVs)

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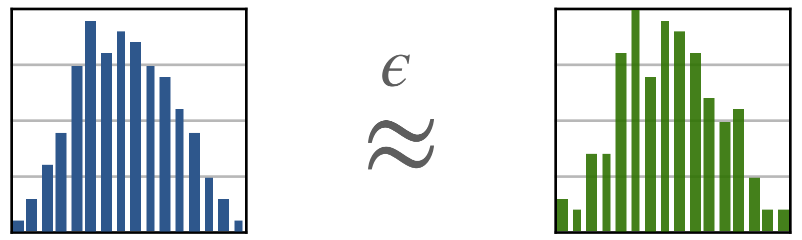
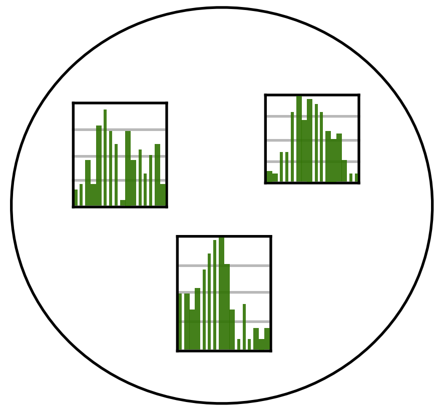
Challenge 1.
Sample Complexity independent from n



Distribution of Samples (from the sum)

Output Distribution

Challenge 2.
Representatives are SIIRVs themselves (proper covering)



However, *in the worst case*, if the terms X_i have:

Collective support of size $\geq$ 4:
The sample complexity scales with the maximum value of the support
[DLS18]

Unbounded support:
The sample complexity scales (polynomially) with n
[DDO+13]

Our results

Warm-up: Addressing Challenge 1 for "nice" unbounded distributions

We focus on a specific type of integer random variables:

$\sum_{i=1}^{n} X_i$ with independent, integer valued terms

&

Assumption 1.

Each term X_i is "nice", i.e.,

1. Unimodal & far from deterministic
2. Modes within a bounded region
3. Bounded fourth central moment

Theorem. Under Assumption 1, the distribution of an unknown SIIRV can be estimated up to error ϵ in statistical distance, using poly$(1/\epsilon)$ independent samples from the sum. (challenge 1)

Moreover, any family of SIIRVs that satisfy Assumption 1, can be ϵ-covered in statistical distance by the union of a collection of $2^{\text{poly}(1/\epsilon)}$ SIIRVs with the set of Discretized Gaussian random variables.

Our results

Main Result: SIIERVs

We focus on a specific type of integer random variables:

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&

Assumption 2.

Each term X_i is belongs in a given “nice” exponential family $\mathscr{E}$ with k parameters.

Remark: we then call the sum an $\mathscr{E}$-SIIERV

Theorem. Under Assumption 2, the distribution of an unknown $\mathscr{E}$-SIIERV can be estimated up to error ϵ in statistical distance, using $\tilde{O}(k/\epsilon^2)$ independent samples from the sum. The output of the learning algorithm is itself an $\mathscr{E}$-SIIERV. (challenge 1)

Moreover, the family of $\mathscr{E}$-SIIERVs admits a proper sparse cover of size $2^{\tilde{O}(k/\epsilon^2)} + n^2 \cdot O(1/\epsilon)^k$. (challenge 2)

Proof Ingredients

- Sparsely covering a "nice" exponential family $\mathscr{E}$. *(case $n = 1$).*

  Geometric properties of polyhedral cones $\Rightarrow$ Bound the range of parameters

  Covering the bounded version of the parameter space $\Rightarrow$ Covering $\mathscr{E}$

- (Proper) structural results for $\mathscr{E}$-SIIERVs.

  Sparse Case (small n): Use sparse covers for $\mathscr{E}$.

  Dense Case (large n): Use appropriate Berry-Esseen type bound, unimodality and continuity of moments.

- Learning $\mathscr{E}$-SIIERVs.

  Given structural results, carefully apply standard methods to design a learning algorithm.

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Thank you!