

Support Recovery for Orthogonal Matching Pursuit: Upper and Lower bounds

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Sparse Linear Regression (SLR)

$$\bar{\mathbf{x}} = \arg \min_{\|\mathbf{x}\|_0 \leq s^*} \left\| \begin{matrix} \mathbf{A} \\ \mathbf{x} \end{matrix} - \begin{matrix} \mathbf{y} \end{matrix} \right\|_2^2$$

- Unconditionally, NP hard. 😞
- Tractable under the assumption of Restricted Strong Convexity (RSC). 😊
- Fundamental quantity capturing hardness :-

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The diagram shows the optimization problem for Sparse Linear Regression. On the left, the expression $\bar{\mathbf{x}} = \arg \min_{\|\mathbf{x}\|_0 \leq s^*}$ is followed by a vertical double line. To the right of this line is a large rectangle labeled \mathbf{A} . To the right of \mathbf{A} is a vertical column of six boxes, the second one from the top is labeled \mathbf{x} . To the right of this column is a minus sign, followed by a vertical column of six boxes, the second one from the top is labeled \mathbf{y} . To the right of this column is another vertical double line, with a 2 above and below it, indicating the L2 norm.

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Sparse Linear Regression (SLR)

$$\bar{x} = \arg \min_{\|x\|_0 \leq s^*} \|Ax - y\|_2^2$$

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- Fundamental quantity capturing hardness :-
 - Standard optimization : Condition number

$$\kappa = \frac{\text{smoothness}}{\text{strong convexity}}$$

- Sparse optimization : Restricted Condition number

$$\tilde{\kappa} = \frac{\text{restricted smoothness}}{\text{restricted strong convexity}}$$

Setup and Goals

We work under the model where

- Observations
- Measurement matrix

$$\mathbf{y} = \mathbf{A} \bar{\mathbf{x}} + \boldsymbol{\eta}$$

- s^* -sparse vector
- Noise

Goals of SLR

- $\hat{\mathbf{x}}$ is a good approximation of $\bar{\mathbf{x}}$ (i.e. $\|\hat{\mathbf{x}} - \bar{\mathbf{x}}\|_2$ is small)
- $\hat{\mathbf{x}}$ is s^* -sparse (i.e. $\|\hat{\mathbf{x}}\|_0 \leq s^*$)

We study SLR under RSC assumption for OMP.

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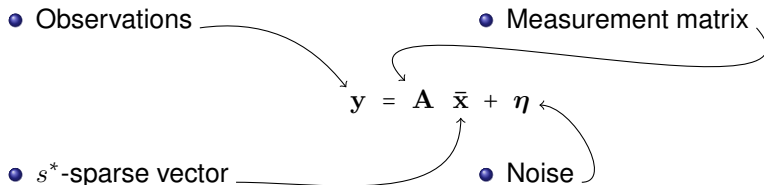
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Goals of SLR

- 1 **Bounding Generalization error/Excess Risk** - $G(\mathbf{x}) := \frac{1}{n} \|\mathbf{A}(\mathbf{x} - \bar{\mathbf{x}})\|_2^2$.
- 2 **Support Recovery** - Recover the support of $\bar{\mathbf{x}}$.

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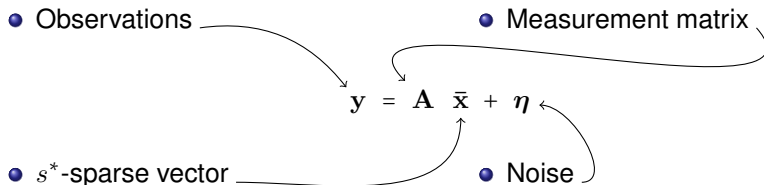
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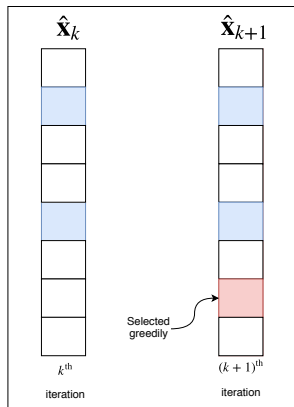
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Orthogonal Matching Pursuit

- Incremental Greedy algorithm
- Popular and easy to implement
- Widely studied in literature



Known results and our contribution

	<u>Upper bound</u>	<u>Lower bound</u>
Known Generalization bound \propto	$\frac{1}{n} \sigma^2 s^* \tilde{\kappa}^2$	$\frac{1}{n} \sigma^2 s^* \tilde{\kappa}$
Our Generalization bound \propto	$\frac{1}{n} \sigma^2 s^* \tilde{\kappa} \log \tilde{\kappa}$	$\frac{1}{n} \sigma^2 s^* \tilde{\kappa}$

Underworld lower bounds (Chandrasekaran et al. 2012)

Support recovery guarantees and the lower bounds

Support Expansion

Known \propto	$s^* \tilde{\kappa}^2$
Our's \propto	$s^* \tilde{\kappa} \log \tilde{\kappa}$

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A key idea

$$f(\mathbf{x}) = \|\mathbf{Ax} - \mathbf{y}\|_2^2$$

- If *any* support is unrecovered, then there is a *large additive decrease*.
- $f(\mathbf{x}) \geq 0 \implies$ support recovery will happen soon.
- Recovery with small support \implies small generalization error.

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