Support Recovery for Orthogonal Matching Pursuit:

Upper and Lower bounds

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Sparse Linear Regression (SLR)

$$\bar{\mathbf{x}} = \arg \min_{\|\mathbf{x}\|_0 \leq s^*} \left\| \mathbf{A}\mathbf{x} - \mathbf{y} \right\|_2^2$$

- Unconditionally, NP hard. ☹
- Tractable under the assumption of Restricted Strong Convexity (RSC). ☺
- Fundamental quantity capturing hardness :-

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  - Standard optimization : Condition number

$$\kappa = \frac{\text{smoothness}}{\text{strong convexity}}$$

  - Sparse optimization : Restricted Condition number

$$\tilde{\kappa} = \frac{\text{restricted smoothness}}{\text{restricted strong convexity}}$$

Setup and Goals

We work under the model where

- Observations
- Measurement matrix

$$\mathbf{y} = \mathbf{A}\bar{\mathbf{x}} + \boldsymbol{\eta}$$

- s^*-sparse vector
- Noise

Goals of SLR

- Bounding Generalization error/Excess Risk – [illegible] $\frac{1}{n}\|A(x - \bar{x})\|_2^2$
- Support Recovery – Recover the support of $\bar{x}$

We study SLR under RSC assumption for OMP.

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- s^*-sparse vector → $\bar{\mathbf{x}}$
- Noise → $\boldsymbol{\eta}$

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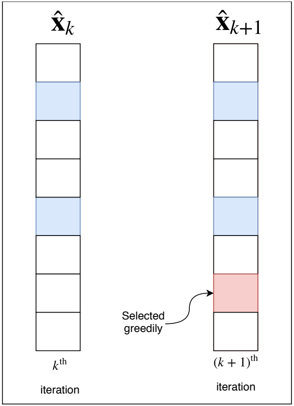
Orthogonal Matching Pursuit

- Incremental Greedy algorithm
- Popular and easy to implement
- Widely studied in literature



Known results and our contribution

	Upper bound	**Lower bound**
Known Generalization bound $\propto$	$\frac{1}{n}\sigma^2 s^* \widetilde{\kappa}^2$	$\frac{1}{n}\sigma^2 s^* \widetilde{\kappa}$
Our Generalization bound $\propto$	$\frac{1}{n}\sigma^2 s^* \widetilde{\kappa} \log \widetilde{\kappa}$	$\frac{1}{n}\sigma^2 s^* \widetilde{\kappa}$

- Unconditional lower bounds for OMP
- Support recovery guarantees

Support Expansion

Known $\propto$	$s^* \widetilde{\kappa}^2$
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A key idea

$$f(\mathbf{x}) = \|\mathbf{A}\mathbf{x} - \mathbf{y}\|_2^2$$

- If *any* support is unrecovered, then there is a *large additive decrease*.
- $f(\mathbf{x}) \geq 0 \implies$ support recovery will happen soon.
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Thank You!