Beyond Online Balanced Descent: An Optimal Algorithm for Smoothed Online Convex Optimization

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Based on joint work with Yiheng Lin, Haoyuan Sun, and Adam Wierman
This talk: how do we design online learning algorithms that adapt to dynamic environments while accounting for switching costs?
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Online Convex Optimization (OCO) with **one-step lookahead** and **switching costs**

An online learner plays a series of rounds against an adaptive adversary. In the \( t \)-th round:

1. The adversary chooses an \( m \)-strongly-convex cost function \( f_t : \mathbb{R}^d \rightarrow \mathbb{R} \geq 0 \).
2. After observing \( f_t \), the learner picks a point \( x_t \in \mathbb{R}^d \).
3. The online learner pays the hitting cost \( f_t(x_t) \) as well as a switching cost \( \frac{1}{2} \| x_t - x_{t-1} \|^2 \) which penalizes the learner for changing its decisions between rounds.
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Online Convex Optimization (OCO) with \textbf{one-step lookahead} and \textbf{switching costs}

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2. After observing $f_t$, the learner picks a point $x_t \in \mathbb{R}^d$.

3. The online learner pays the \textbf{hitting cost} $f_t(x_t)$ as well as a \textbf{switching cost} $\frac{1}{2} \| x_t - x_{t-1} \|^2_2$ which penalizes the learner for changing its decisions between rounds.
Competitive Ratio = \sup_{f_1, \ldots, f_T} \frac{\sum_{t=1}^{T} f_t(x_t) + \frac{1}{2} \|x_t - x_{t-1}\|^2}{\min_{x_1, \ldots, x_T} \sum_{t=1}^{T} f_t(x_t) + \frac{1}{2} \|x_t - x_{t-1}\|^2}.

\text{Dynamic optimal solution}
Online Gradient Descent (OGD)

\[ x_{t-1} \]

\[ x_t \]

\[ v_t = \arg\min_{x_t} f_t(x) \]
Online Balanced Descent (OBD)

Key idea 1: Always project onto level sets (same hitting cost but less switching cost!).
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Key idea 2: Pick step size so that switching costs and hitting costs are roughly equal.
Theorem (Goel, Lin, Sun, Wierman ’19)

Suppose the hitting cost functions are $m$-strongly convex with respect to the $\ell_2$ norm and the switching cost is given by $c(x_t, x_{t-1}) = \frac{1}{2} \|x_t - x_{t-1}\|_2^2$. Any online algorithm must have a competitive ratio at least $\frac{1}{2} \left(1 + \sqrt{1 + \frac{4}{m}}\right)$. Furthermore, this lower bound is not achieved by OBD.
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Thanks for listening! See poster #50 at 5pm today.

Connections to statistics and control: An Online algorithm for Smoothed Regression and LQR Control [Goel and Wierman, AISTATS’19]

Non-convex cost functions: Online Optimization with Predictions and Non-convex Losses [Lin, Goel, and Wierman arXiv 1911.03827]