Beyond Online Balanced Descent: An Optimal Algorithm for Smoothed Online Convex Optimization

Gautam Goel

Based on joint work with Yiheng Lin, Haoyuan Sun, and Adam Wierman
Online learning has traditionally focused on designing algorithms with low *static regret* against the best *fixed* action.

Many examples: multi-armed bandits, k-experts, Online Convex Optimization (OCO)…
This talk: how do we design online learning algorithms that adapt to dynamic environments while accounting for switching costs?
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Smoothed Online Convex Optimization (SOCO)

An online learner plays a series of rounds against an adaptive adversary. In the $t$-th round:

1. The adversary chooses an $m$-strongly-convex cost function $f_t : \mathbb{R}^d \to \mathbb{R} \geq 0$.
2. After observing $f_t$, the learner picks a point $x_t \in \mathbb{R}^d$.
3. The online learner pays the hitting cost $f_t(x_t)$ as well as a switching cost $\frac{1}{2} \| x_t - x_{t-1} \|^2$ which penalizes the learner for changing its decisions between rounds.
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3. The online learner pays the **hitting cost** $f_t(x_t)$ as well as a **switching cost**
   $\frac{1}{2} \| x_t - x_{t-1} \|_2^2$ which penalizes the learner for changing its decisions between rounds.
Competitive Ratio = \( \sup_{f_1, \ldots, f_T} \frac{\sum_{t=1}^{T} f_t(x_t) + \frac{1}{2} \|x_t - x_{t-1}\|^2}{\min_{x_1, \ldots, x_T} \sum_{t=1}^{T} f_t(x_t) + \frac{1}{2} \|x_t - x_{t-1}\|^2} \).

Dynamic optimal solution
Online Gradient Descent (OGD)

\[ v_t = \text{argmin}_{x \in \mathbb{R}^d} f_t(x) \]
Online Balanced Descent (OBD)

Key idea 1: Always project onto level sets (same hitting cost but less switching cost!).
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**Key idea 1:** Always project onto level sets (same hitting cost but less switching cost!).

**Key idea 2:** Pick step size so that switching costs and hitting costs are roughly equal.
Theorem (Goel, Lin, Sun, Wierman ’19)

Suppose the hitting cost functions are $m$-strongly convex with respect to the $\ell_2$ norm and the switching cost is given by $c(x_t, x_{t-1}) = \frac{1}{2} \|x_t - x_{t-1}\|_2^2$. Any online algorithm must have a competitive ratio at least $\frac{1}{2} \left( 1 + \sqrt{1 + \frac{4}{m}} \right)$. Furthermore, this lower bound is not achieved by OBD.
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Thanks for listening! See poster #50 on Wednesday at 7pm.

Connections to statistics and control: An Online algorithm for Smoothed Regression and LQR Control [Goel and Wierman, AISTATS’19]

Non-convex cost functions: Online Optimization with Predictions and Non-convex Losses [Lin, Goel, and Wierman arXiv 1911.03827]