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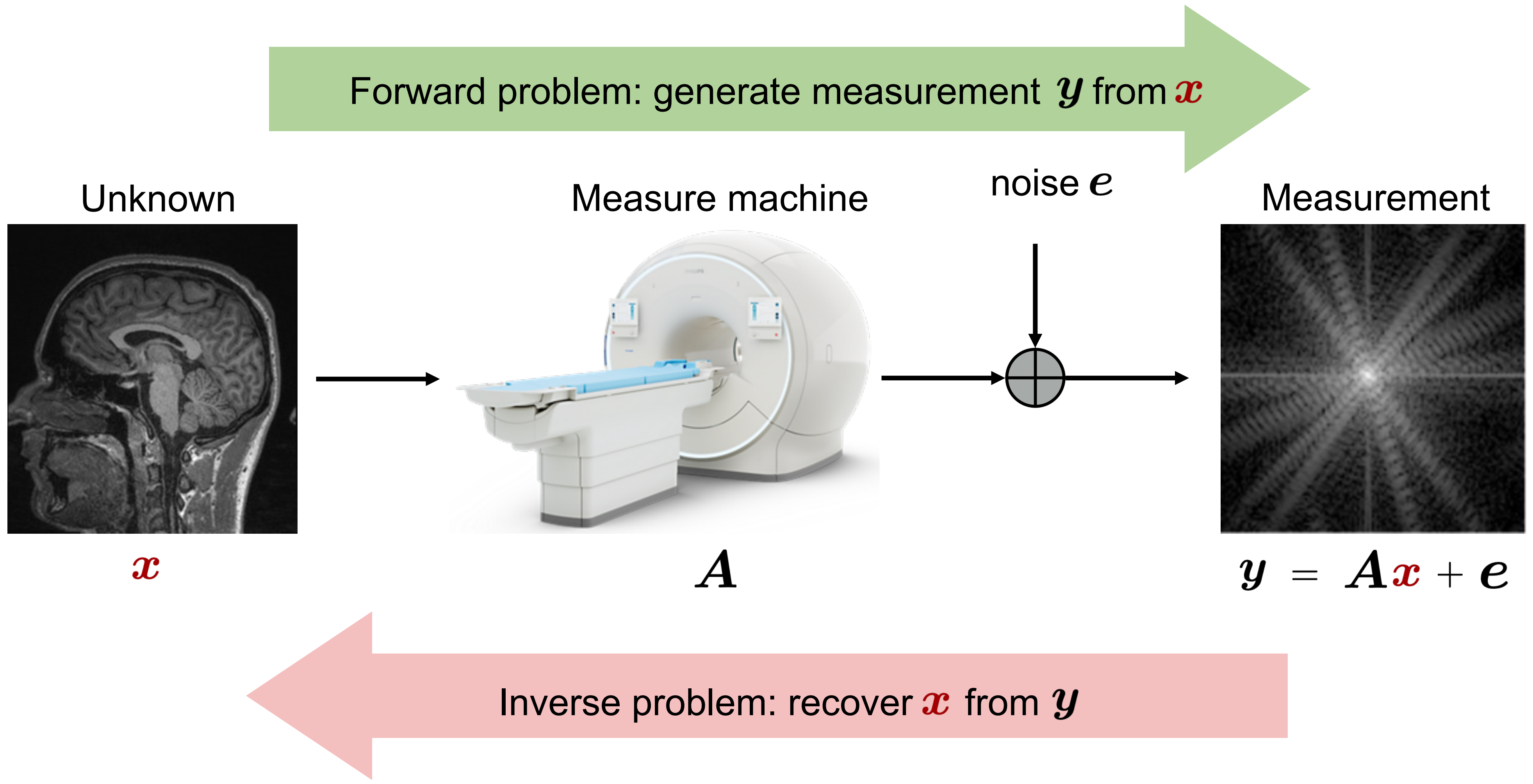
Recovery Analysis for Plug-and-Play Priors using the Restricted Eigenvalue Condition

Jiaming Liu

Joint work with M. Salman Asif,
Brendt Wohlberg and Ulugbek S. Kamilov

Connect with me:
Email: jiaming.liu@wustl.edu

Most imaging problems can be formulated as inverse problems



Classic approach transforms the inverse problem to a regularized optimization

Inverse problem: $\boldsymbol{y} = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{e}$

Regularized optimization: $\underset{\boldsymbol{x}}{\arg\min} \; \{ \; g(\boldsymbol{x}) \; + \; h(\boldsymbol{x}) \; \}$

Data-fidelity　Regularizer

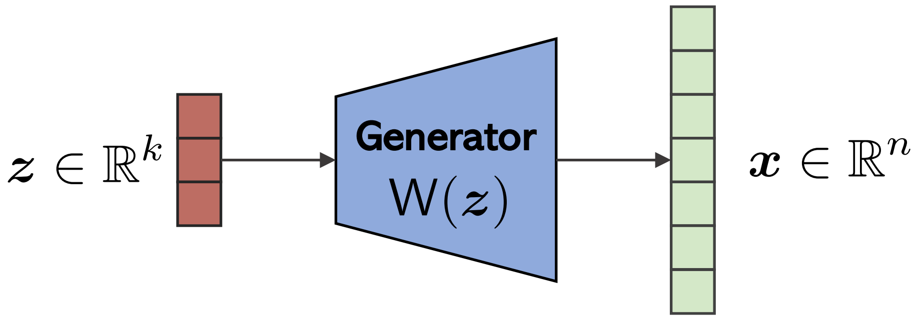
$$g(\boldsymbol{x}) = \|\boldsymbol{A}\boldsymbol{x} - \boldsymbol{y}\|_2^2$$

Least-square loss

Total variation (TV)

Compressed sensing using generative models (CSGM)

Idea: **Pre-train** a generative model on a dataset of images



CSGM: Generative priors for solving inverse problems [Bora et al. 2017]

$$\min_{\boldsymbol{z}\in\mathbb{R}^k} = \frac{1}{2}\|\boldsymbol{y} - \boldsymbol{A}\mathsf{W}(\boldsymbol{z})\|_2^2$$

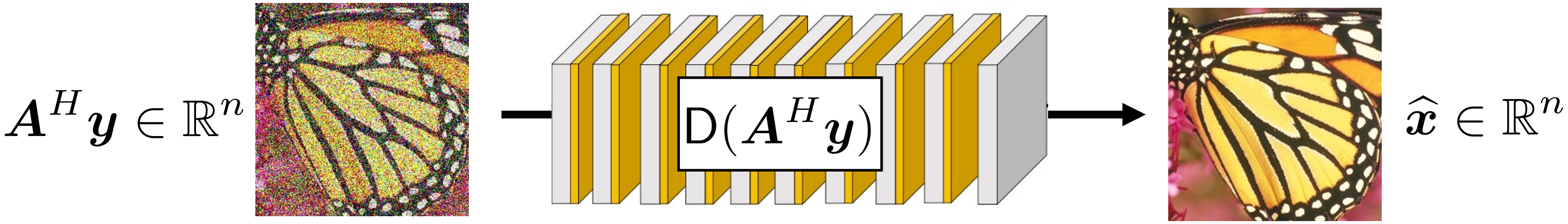
with an appropriate choice of A satisfying the *set-restricted eigenvalue condition* (S-REC) over the range of the generator

$$\|\boldsymbol{A}\boldsymbol{x} - \boldsymbol{A}\boldsymbol{z}\|_2^2 \geq \mu\|\boldsymbol{x} - \boldsymbol{z}\|_2^2 - \eta \quad \forall \boldsymbol{x}, \boldsymbol{z} \in \mathbb{R}^n$$

Where $\mu > 0$ and $\eta \geq 0$

Plug-and-play priors (PnP) and Regularization by denoising (RED)

Idea: **Pre-train** an artifact removing convolutional neural network (CNN) on a dataset of images [Liu et al. 2020]



PnP-PGM: replacing $\mathrm{prox}_{\gamma h}$ within the proximal gradient method (PGM) by denoiser or artifact removal CNN [Venkatakrishnan et al. 2013]

$\boldsymbol{x}^k \leftarrow \mathsf{D}(\boldsymbol{z}^k)$ Reduce image artifacts

$\boldsymbol{z}^{k-1} \leftarrow \boldsymbol{x}^{k-1} - \gamma \nabla g(\boldsymbol{x}^{k-1})$ Improve data fit

PnP-PGM can be rewritten within one line with a transfer operator T

$$\boldsymbol{x}^k = \mathsf{T}(\boldsymbol{x}^{k-1}) \qquad \text{with} \qquad \mathsf{T} := \mathsf{D}(\mathsf{I} - \gamma \nabla g)$$

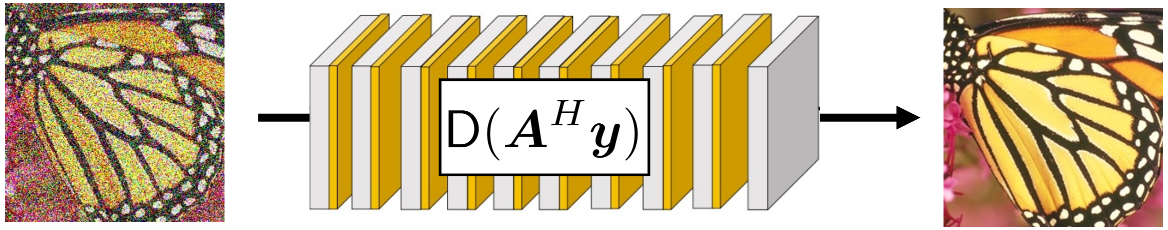
When the residual of D is Lipschitz continuous, PnP-PGM converges to a point in the fixed-point set of the operator T [Ryu et al. 2019]

$$\mathsf{Fix}(\mathsf{T}) := \{\boldsymbol{x} \in \mathbb{R}^n : \boldsymbol{x} = \mathsf{T}(\boldsymbol{x})\}$$

Plug-and-play priors (PnP) and Regularization by denoising (RED)

Idea: **Pre-train** an artifact removing convolutional neural network (CNN) on a dataset of images [Liu et al. 2020]

$\boldsymbol{A}^H\boldsymbol{y} \in \mathbb{R}^n$



$\widehat{\boldsymbol{x}} \in \mathbb{R}^n$

SD-RED: the steepest descent variant of RED cam ne summarized as [Romano et al. 2017]

$$\boldsymbol{x}^k = \boldsymbol{x}^{k-1} - \gamma \mathsf{G}(\boldsymbol{x}^{k-1}) \qquad \text{with} \qquad \mathsf{G}(\boldsymbol{x}) = \nabla g(\boldsymbol{x}) + \tau(\boldsymbol{x} - \mathsf{D}(\boldsymbol{x}))$$

"gradient" descent

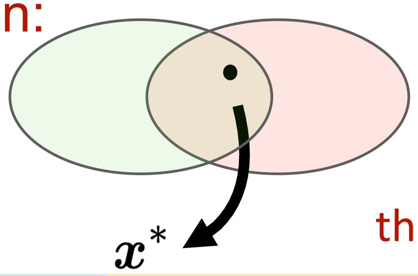
Improve data fit

Reduce image artifact

Optimal condition: suppose there exists a vector that satisfy

$$\boldsymbol{x}^* \in \mathsf{Zer}(\nabla g) \cap \mathsf{Fix}(\mathsf{D}) \quad \text{with} \quad \mathsf{Zer}(\nabla g) := \{\boldsymbol{x} \in \mathbb{R}^n : \nabla g = \boldsymbol{0}\} \quad \mathsf{Fix}(\mathsf{D}) := \{\boldsymbol{x} \in \mathbb{R}^n : \boldsymbol{x} = \mathsf{D}(\boldsymbol{x})\}$$

Intuitive explanation:



$$\Longrightarrow \boldsymbol{0} = \nabla g(\boldsymbol{x}^*) + \tau(\boldsymbol{x}^* - \mathsf{D}(\boldsymbol{x}^*))$$

the algorithm can converge to the fixed-point $\boldsymbol{x}^*$

Recovery Analysis for PnP and RED

Latent space generative model

$$\min_{\boldsymbol{z}\in\mathbb{R}^k} = \frac{1}{2}\|\boldsymbol{y} - \boldsymbol{A}\mathsf{W}(\boldsymbol{z})\|_2^2$$

- Pre-trained generative model.
- Domain specific, can recover images on the range of generative model.
- Nonconvex projections onto the range of a generative model.

PnP/RED

$$\boldsymbol{x}^k = \mathsf{T}(\boldsymbol{x}^{k-1}) \quad \text{with} \quad \mathsf{T} := \mathsf{D}(\mathsf{I} - \gamma\nabla g)$$

$$\boldsymbol{x}^k = \boldsymbol{x}^{k-1} - \gamma\mathsf{G}(\boldsymbol{x}^{k-1}) \quad \text{with} \quad \mathsf{G} = \nabla g + \tau(\mathsf{I} - \mathsf{D})$$

- Pre-trained denoiser or artifact remove operator.
- Leverage any off-the-shelf regularizer.
- Lack theoretical recovery guarantees available for CSGM.

Contributions of this paper:

- We first establish recovery bounds for PnP and address the relationship between the solutions of PnP and RED, under a set of sufficient conditions.
- Numerical analysis of PnP/RED and CSGM.

Recovery Analysis for PnP and RED: Numerical evaluation

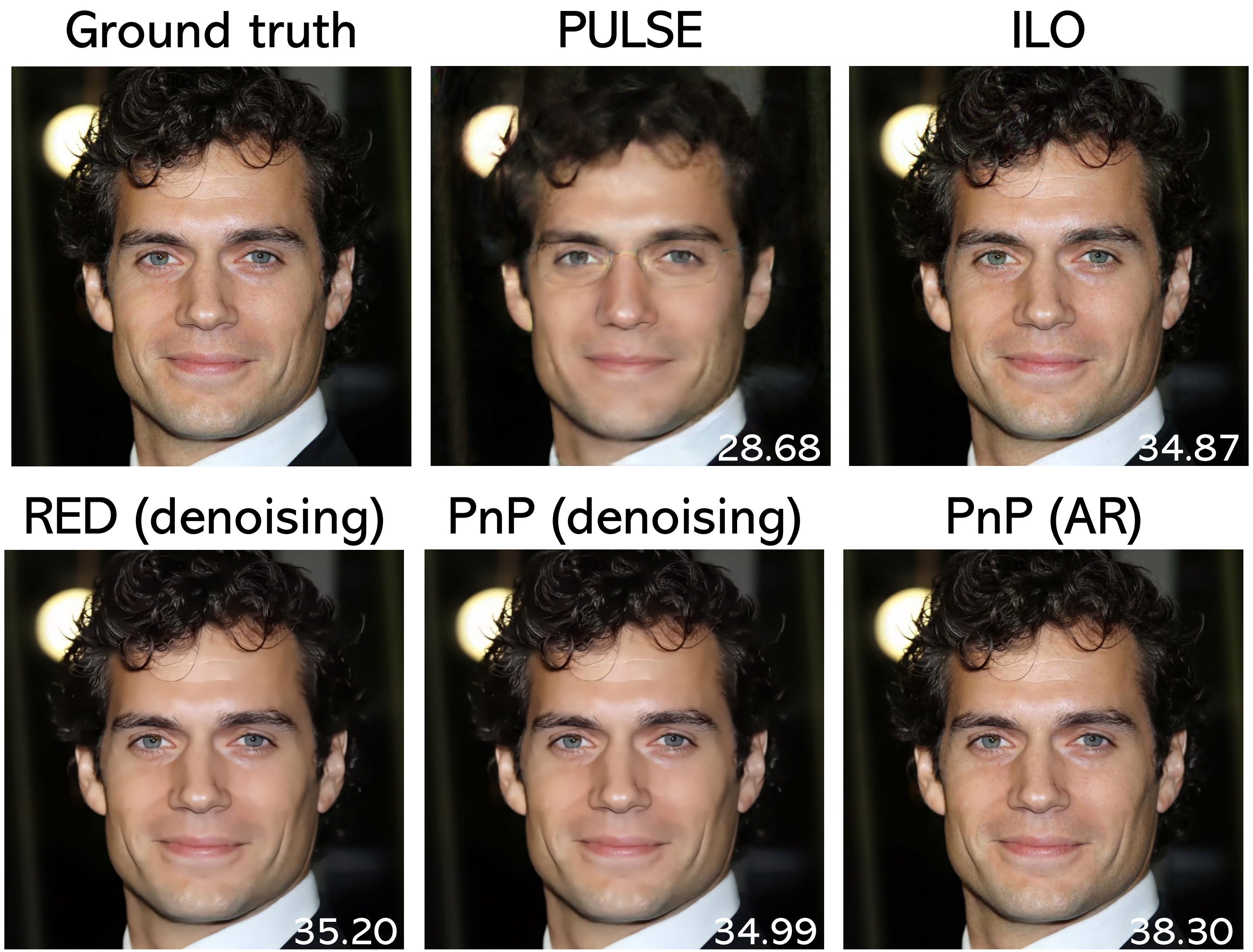


Table 3: Average PSNR (dB) values for several algorithms on test images from CelebA HQ.

Method \ CS Ratio	10%	20%	30%
TV	32.13	35.24	37.41
PULSE [34]	27.45	29.98	33.06
ILO [35]	36.15	40.98	43.46
RED (denoising)	35.46	41.59	45.65
PnP (denoising)	35.61	41.51	45.71
PnP (AR)	**39.19**	**44.20**	**48.66**

- AR is better than AWGN (expected!)
- RED is nearly equivalent of PnP (somehow surprising!)
- PnP(AR) is competitive with PULSE and ILO using StyleGan2 (surprising!)

Scaling strategy for PnP is adopted from [Xiao et al. 20]

Recovery Analysis for PnP and RED: Main results

Theorem 1. *Run PnP-PGM for $t \geq 1$ iterations under Assumptions 1-2 for the regularized problem with no noise and $\boldsymbol{x}^* \in \mathsf{Zer}(\mathsf{R})$. Then, the sequence $\boldsymbol{x}^t$ generated by PnP-PGM satisfies*

$$\|\boldsymbol{x}^t - \boldsymbol{x}^*\|_2 \leq c\|\boldsymbol{x}^{t-1} - \boldsymbol{x}^*\|_2 \leq c^t\|\boldsymbol{x}^0 - \boldsymbol{x}^*\|_2 \, , \tag{1}$$

where $\boldsymbol{x}^0 \in \mathbb{R}^n$ and $c := (1+\alpha)\max\{|1-\gamma\mu|, |1-\gamma\lambda|\}$ with $\lambda := \lambda_{\mathsf{max}}(\boldsymbol{A}^\mathsf{T}\boldsymbol{A})$.

Theorem 1 extends the theoretical analysis of PnP in [Ryu et al. 2019] by showing convergence to the true solution x∗ instead of the fixed points Fix(T) of T. Note that the condition x0 in the range of D can be easily enforced by simply passing any initial image through the operator D.

Recovery Analysis for PnP and RED: Main results

Theorem 2. *Run PnP-PGM for $t \geq 1$ iterations under Assumptions 1-2 for the regularized problem with $\boldsymbol{x}^* \in \mathbb{R}^n$ and $\boldsymbol{e} \in \mathbb{R}^m$. Then, the sequence $\boldsymbol{x}^t$ generated by PnP-PGM satisfies*

$$\|\boldsymbol{x}^t - \boldsymbol{x}^*\|_2 \leq c\|\boldsymbol{x}^{t-1} - \boldsymbol{x}^*\|_2 + \varepsilon \leq c^t\|\boldsymbol{x}^0 - \boldsymbol{x}^*\|_2 + \frac{\varepsilon(1-c^t)}{(1-c)}\,, \tag{2}$$

where

$$\varepsilon := (1+c)\left[\left(1+2\sqrt{\lambda/\mu}\right)\|\boldsymbol{x}^* - \mathsf{proj}_{\mathsf{Zer(R)}}(\boldsymbol{x}^*)\|_2 + 2/\sqrt{\mu}\|\boldsymbol{e}\|_2 + \delta(1+1/\alpha)\right] \tag{3}$$

and $c := (1+\alpha)\,\mathsf{max}\{|1-\gamma\mu|, |1-\gamma\lambda|\}$ *with* $\lambda := \lambda_{\mathsf{max}}(\boldsymbol{A}^\mathsf{T}\boldsymbol{A})$.

Theorem 2 extends Theorem 1 by allowing x∗ to be outside of Fix(D) and extends the analysis in [Bora et al. 2017] by considering operators D that do not necessarily project onto the range of a generative model.

Recovery Analysis for PnP and RED: Main results

Theorem 2. *Run PnP-PGM for $t \geq 1$ iterations under Assumptions 1-2 for the regularized problem with $\boldsymbol{x}^* \in \mathbb{R}^n$ and $\boldsymbol{e} \in \mathbb{R}^m$. Then, the sequence $\boldsymbol{x}^t$ generated by PnP-PGM satisfies*

$$\|\boldsymbol{x}^t - \boldsymbol{x}^*\|_2 \leq c\|\boldsymbol{x}^{t-1} - \boldsymbol{x}^*\|_2 + \varepsilon \leq c^t\|\boldsymbol{x}^0 - \boldsymbol{x}^*\|_2 + \frac{\varepsilon(1-c^t)}{(1-c)}\ , \tag{2}$$

where

$$\varepsilon := (1+c)\left[\left(1+2\sqrt{\lambda/\mu}\right)\|\boldsymbol{x}^* - \mathsf{proj}_{\mathsf{Zer(R)}}(\boldsymbol{x}^*)\|_2 + 2/\sqrt{\mu}\|\boldsymbol{e}\|_2 + \delta(1+1/\alpha)\right] \tag{3}$$

and $c := (1+\alpha)\,\mathsf{max}\{|1-\gamma\mu|, |1-\gamma\lambda|\}$ *with* $\lambda := \lambda_{\mathsf{max}}(\boldsymbol{A}^\mathsf{T}\boldsymbol{A})$.

Theorem 3. *Suppose that Assumptions 1-3 are satisfied and that* $\mathsf{Zer}(\nabla g) \cap \mathsf{Zer}(\mathsf{R}) \neq \varnothing$, *then PnP and RED have the same set of solutions:* $\mathsf{Fix}(\mathsf{T}) = \mathsf{Zer}(\mathsf{G})$.

Proof Technology

Assumption 1. *The residual* $\mathsf{R} := \mathsf{I} - \mathsf{D}$ *of the operator* D *is bounded by* δ *and Lipschitz continuous with constant* $\alpha > 0$.

$$\|\mathsf{R}(\boldsymbol{x})\|_2 \leq \delta \quad \textit{and} \quad \|\mathsf{R}(\boldsymbol{x}) - \mathsf{R}(\boldsymbol{z})\|_2 \leq \alpha\|\boldsymbol{x} - \boldsymbol{z}\|_2, \quad \forall \boldsymbol{x}, \boldsymbol{z} \in \mathbb{R}^n \ .$$

Assumption 2. *The measurement operator* $\boldsymbol{A} \in \mathbb{R}^{m \times n}$ *satisfies the set-restricted eigenvalue condition (S-REC) over* $\mathsf{Im}(\mathsf{D}) \subseteq \mathbb{R}^n$ *with* $\mu > 0$, *which can be written as*

$$\|\boldsymbol{A}\boldsymbol{x} - \boldsymbol{A}\boldsymbol{z}\|_2^2 \geq \mu\|\boldsymbol{x} - \boldsymbol{z}\|_2^2, \quad \forall \boldsymbol{x}, \boldsymbol{z} \in \mathsf{Im}(\mathsf{D}) \ .$$

Assumption 3. *The denoiser* D *is nonexpansive*

$$\|\mathsf{D}(\boldsymbol{x}) - \mathsf{D}(\boldsymbol{z})\|_2 \leq \|\boldsymbol{x} - \boldsymbol{z}\|_2 \quad \forall \boldsymbol{x}, \boldsymbol{z} \in \mathbb{R}^n \ .$$

- The α-Lipschitz continuity of R can be enforced by using any of the recent techniques for training Lipschitz constrained deep neural nets [Ryu et al. 2019]

- The S-REC in Assumption 2 was adopted from the corresponding assumption for CSGM [Bora et al. 2017]

Conclusion and Broader impact

- ✓ We address the theoretical gap between PnP/RED and CSGM for solving inverse problems
- ✓ Our theoretical results provide a new type of convergence for PnP-PGM by showing convergence relative to the true solution.
- ✓ We show full equivalence of PnP and RED under some explicit conditions on the inverse problem
- ✓ We provide additional evidence on the suboptimality of AWGN denoisers compared to artifact-removal operators
- ✓ Potential applications (e.g., computational microscopy, computerized tomography, medical imaging, and image restoration).

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