Regularization properties of adversarially-trained linear regression

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[*]**Presenting**

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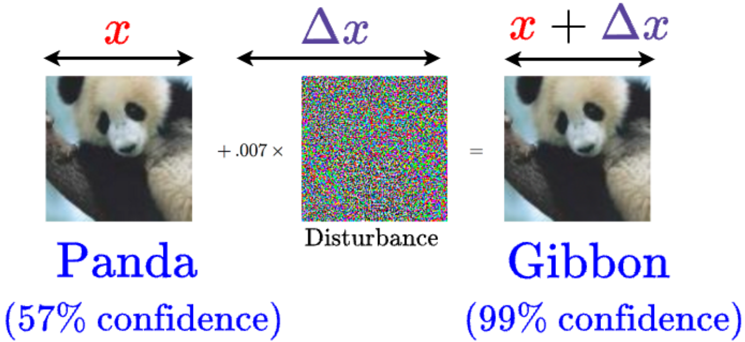
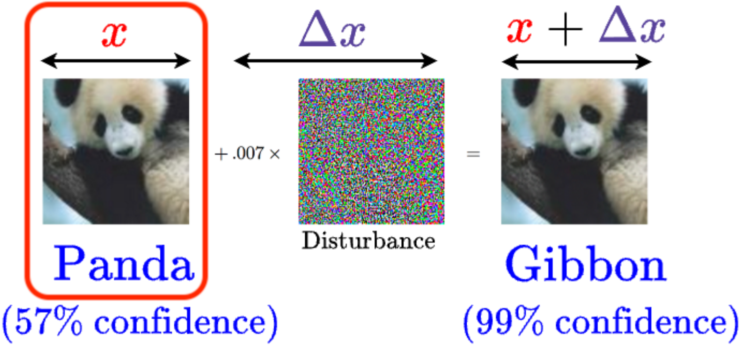
Adversarial attacks



Explaining and Harnessing Adversarial Examples
I. J. Goodfellow, J. Shlens, C. Szegedy
ICLR (2015)

Adversarial attacks



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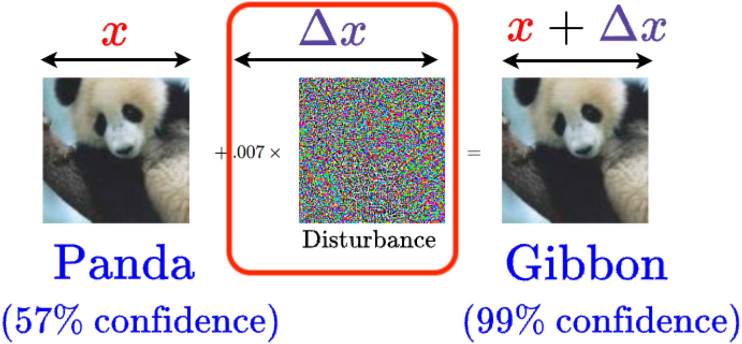

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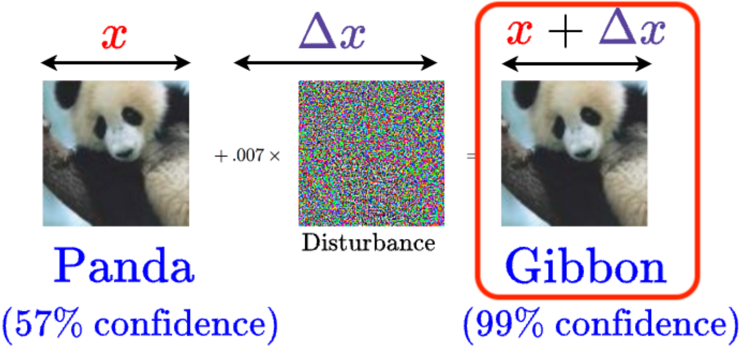

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Adversarial training: *Each training sample is modified by an adversary.*

Adversarially-trained linear regression

- **Linear regression:**

$$\min_{\beta} \sum_{i=1}^{\#train} (y_i - \beta^\top x_i)^2$$

Adversarially-trained linear regression

- **Linear regression:**

$$\min_{\beta} \sum_{i=1}^{\#train} (\underbrace{y_i}_{\text{observed}} - \underbrace{\beta^\top x_i}_{\text{linear prediction}})^2$$

Adversarially-trained linear regression

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It can be ***rewritten*** *as:*

$$\sum_{i=1}^{\#train} \left(|y_i - x_i^\mathsf{T} \beta| + \delta \|\beta\|_* \right)^2$$

where $\| \cdot \|_*$ *is the* ***dual norm****.*

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It can be **rewritten** *as:*

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Similarities with Lasso

- **ℓ_∞-adversarial attacks:**

$$\sum_{i=1}^{\#train} \left(|y_i - x_i^\mathsf{T} \beta| + \delta \|\beta\|_1 \right)^2$$

- **Lasso:**

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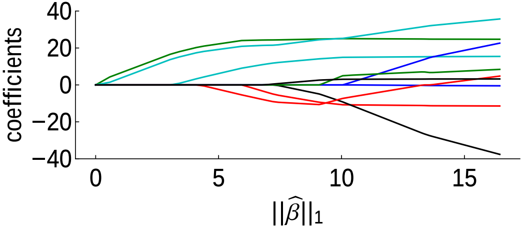
#1. **Map** $\lambda \leftrightarrow \delta$ for which they yield the **same result.**

#2. **More parameters than data**: abrupt transition into interpolation.

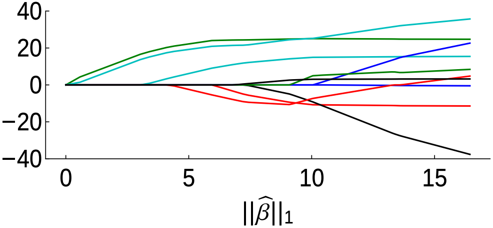
#3. **Optimal choice** of δ independent on noise level.

1. Equivalence with Lasso

Map $\lambda \leftrightarrow \delta$ for which they yield the **same result.**



Lasso.



ℓ_∞-adv. training.

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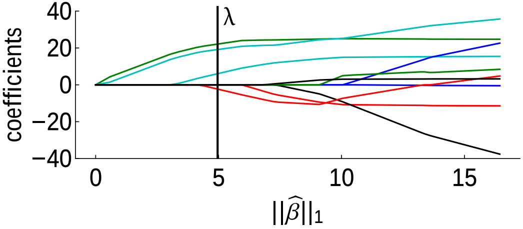

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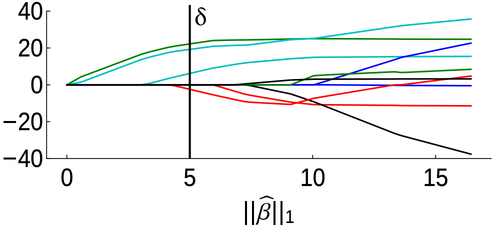

ℓ_∞-adv. training.

The that yield the **same result** are **not** necessarily the same, i.e.: $\delta \neq \lambda$

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Lasso: transition **only in the limit**

$$\lambda \to 0^+ \Rightarrow \text{Mean square error} \to 0$$

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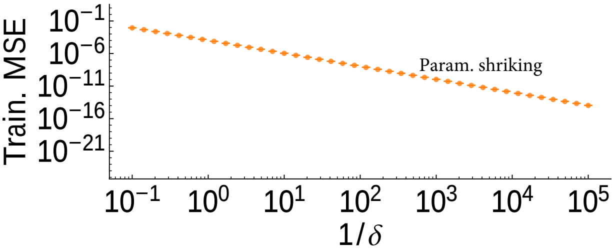

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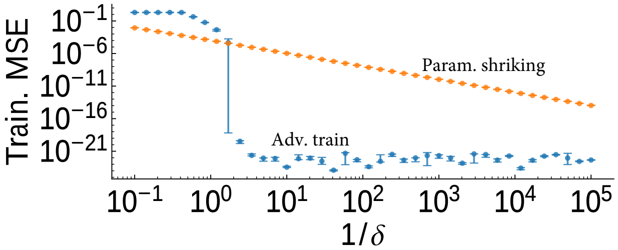

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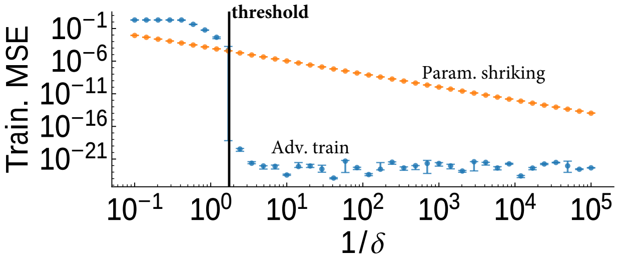

2. Equivalence with minimum norm interpolator

For $\delta \in (0, \mathrm{threshold}]$, the minimum-norm interpolator is the solution to adversarial training.

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Relevance

Connect **adversarial training** with **double descent** and **benign overfitting**

3. Invariance to noise levels

To obtain near-oracle performance.

- *Lasso:*

$$\lambda \propto \sigma \sqrt{\log(\#\textit{params}) / \#\textit{train}}$$

- ℓ_∞-*adversarial attack*:

$$\delta \propto \quad \sqrt{\log(\#\textit{params}) / \#\textit{train}}$$

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To obtain near-oracle performance.

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Data model

$$y = \underbrace{x^\top \beta^*}_{\text{signal}} + \underbrace{\sigma}_{\text{noise std.}} \varepsilon.$$

arXiv:2310.10807

- ℓ_2**-adv.** attacks and **ridge regression.**

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- Generalization to **other loss** functions

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- Generalization to **other loss** functions
- Connection to **robust regression** and $\sqrt{\text{Lasso}}$.