Global Lyapunov functions: a long-standing open problem in mathematics, with symbolic transformers

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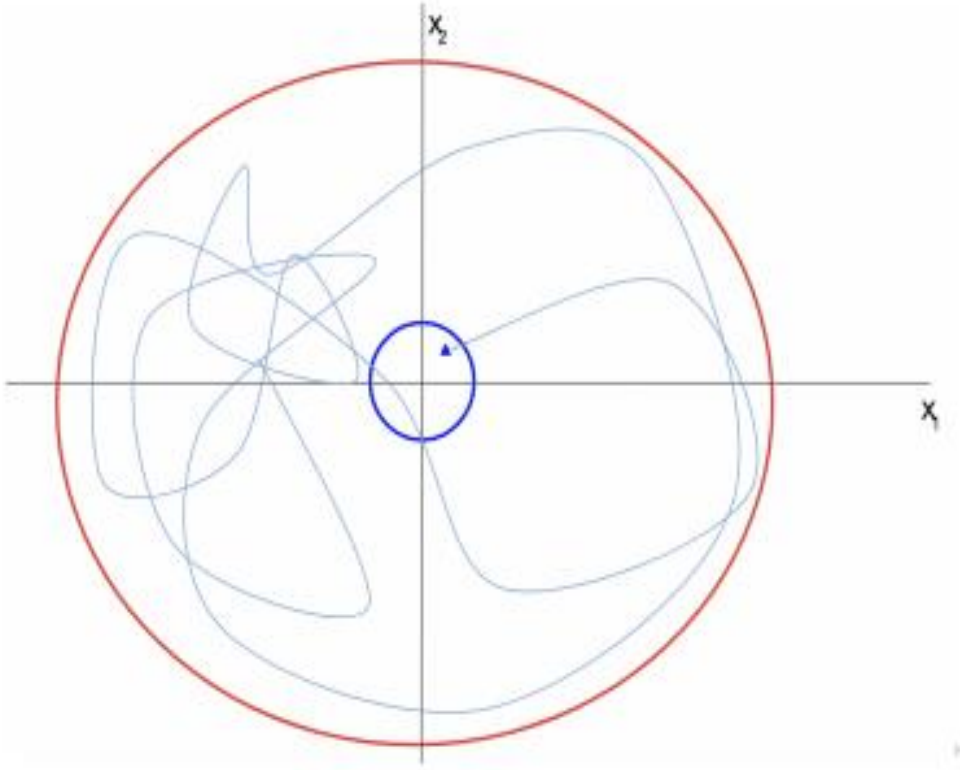
Global stability of dynamical systems

- A dynamical system
  $\dot{x} = f(x), \quad x \in \mathbb{R}^n, \quad f \in C^1(\mathbb{R}^n)$
  is said to be globally stable around an equilibrium if its solutions never diverge to infinity
- A solution that starts close to an equilibrium, always stays close to the equilibrium

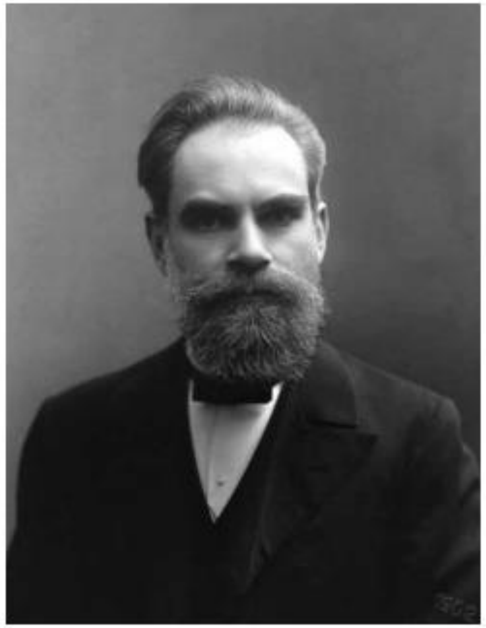


Lyapunov functions

- In 1892, Alexander Lyapunov showed that global stability was achieved if a function V could be found that verified three properties

$$V(x) > V(x_0)$$
$$\lim_{|x| \to +\infty} V(x) = +\infty$$
$$\nabla V(x) . f(x) \leq 0$$

- This is a strong result, but Lyapunov provided no method construct V
- 132 years after Lyapunov, no method is known, except in the simplest cases (polynomials of small degree)

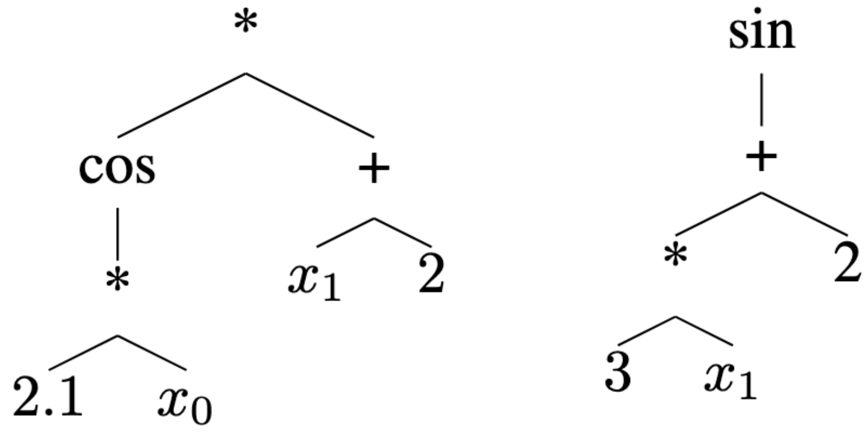


A. Lyapunov
(1857-1918)

Transformers for Lyapunov functions

- Train sequence to sequence transformers to predict Lyapunov functions V, from systems f
- From generated pairs of f and V
- Tokenized as “enumerated trees”
- Using beam search

$$\begin{cases} \dot{x}_0 = \cos(2.1x_0)(x_1 + 2) \\ \dot{x}_1 = \sin(3x_1 + 2) \end{cases}$$



[*, cos, *, 2.1, x_0, +, x_1, 2] [sin, +, *, 3, x_1, 2]

Backward generation

- Training examples are “backward generated”
- Sample a random Lyapunov function V, such that
  - $V(x) > V(x_0)$
  - $\lim_{|x| \to +} V(x) = +\infty$
- Generate an associated system f, verifying
  - $\nabla V(x). f(x) \leq 0$

- So that V and f are as generic as possible
- And the generating process is very hard to reverse

Forward test sets

- Randomly sampling problems, and computing solutions
- Only possible in easy cases (small polynomials)
- Used as test baselines

Main findings

- Models trained on backwards dataset can discover the Lyapunov functions of forward-generated test sets
- The reverse is not true

Backward datasets	FLyap	FBarr	Forward datasets	BPoly
BPoly (polynomial)	**73**	35	FBarr (barrier)	15
BNonPoly (non-poly)	**75**	24	FLyap (Lyapunov)	10

Table 3: **Out-of-domain accuracy of models**. Beam size 50. Columns are the test sets.

Main findings

- Priming: adding a very small number of forward-generated (easy) examples to the training set greatly improves performance

Forward datasets	Examples added (1M in training set)	FLyap	FBarr
No mixture	0	73	35
FBarr	30	75	61
	300	83	89
	3,000	85	93
	30,000	89	95
FLyap	10	75	25
	100	82	29
	1,000	83	37
	10,000	86	38

Main findings

- On a sample of random systems, not necessarily globally stable, we achieve 10-12% accuracy (vs a few % for the best alternatives)

	Sample	SOSTOOL	Existing AI methods			Forward model	Backward model	
Test set	size	`findlyap`	Fossil 2	ANLC	LyzNet	FBarr	PolyMixture	NonPolyMixture
Poly3	65,215	1.1	0.9	0.6	4.3	11.7	**11.8**	11.2
Poly5	60,412	0.7	0.3	0.2	2.1	8.0	**10.1**	9.9
NonPoly	19,746	-	1.0	0.6	3.5	-	-	**12.7**

Table 6: **Discovering Lyapunov comparison for random systems**. Beam size 50. PolyMixture is BPoly + 500 FBarr. NonPolyMixture is BNonPoly + BPoly + 500 FBarr.