Sample-Efficient Geometry Reconstruction from Euclidean Distances using Non-Convex Optimization

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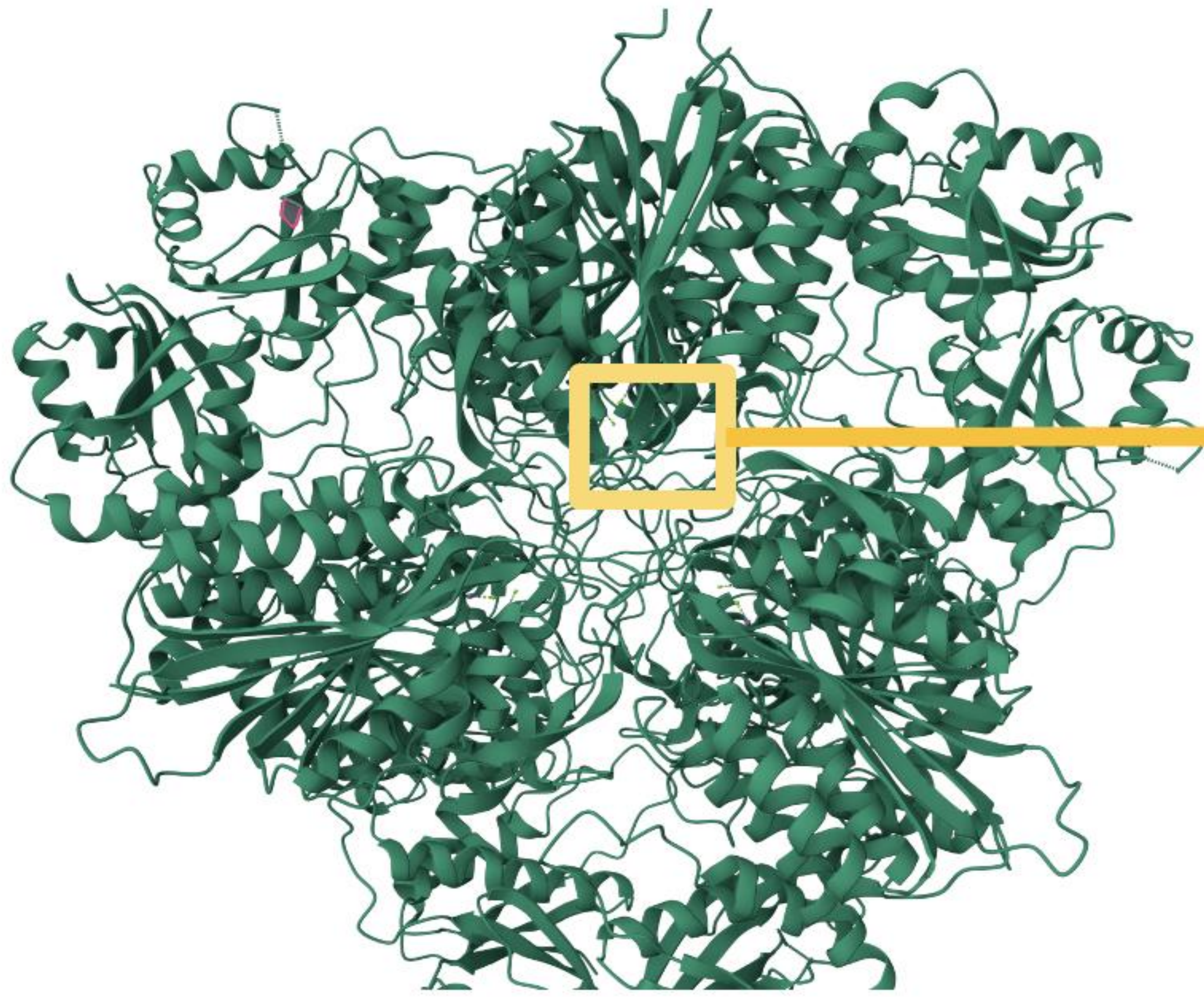
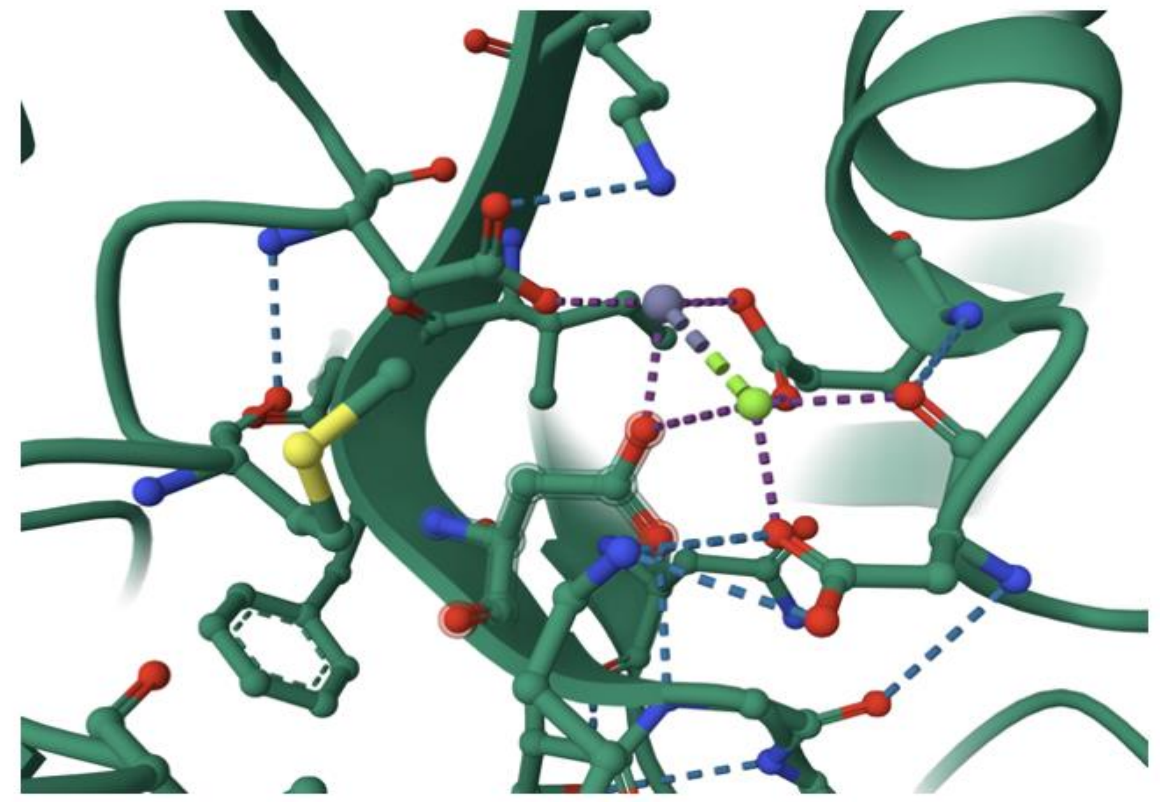
Problem





D_Ω

0	?	?	4
?	0	?	?
?	?	0	2
4	?	2	0

- Given $|\boldsymbol{\Omega}| = \boldsymbol{m} < \boldsymbol{n}(\boldsymbol{n}-\mathbf{1})/\mathbf{2}$ partial distances $\mathbf{d}_{ij}^2 = \|\mathbf{p}_i - \mathbf{p}_j\|^2 = \mathbf{p}_i^\top \mathbf{p}_i + \mathbf{p}_j^\top \mathbf{p}_j - 2\mathbf{p}_i^\top \mathbf{p}_j$,

  $(\boldsymbol{i}, \boldsymbol{j}) \in \mathbf{\Omega}$, (set $\mathbf{\Omega}$ drawn uniformly w/o replacement) between (unknown) points $\boldsymbol{P} = [\boldsymbol{p}_1, \boldsymbol{p}_2, \ldots \boldsymbol{p}_n] \in \mathbb{R}^{r \times n}$
- In a compact form, the distance matrix $\boldsymbol{D} = \mathbf{1}\,\mathrm{diag}(\boldsymbol{P}\boldsymbol{P}^T)^T + \mathrm{diag}(\boldsymbol{P}\boldsymbol{P}^T)\,\mathbf{1}^T - 2\boldsymbol{P}\boldsymbol{P}^T$
- For Gram matrix $\boldsymbol{X} = \boldsymbol{P}^T\boldsymbol{P}$, $\boldsymbol{X}_{i,i} + \boldsymbol{X}_{j,j} - 2\boldsymbol{X}_{i,j} = \boldsymbol{D}_{i,j}\ \forall i, j$, $(\boldsymbol{i}, \boldsymbol{j}) \in \mathbf{\Omega}$

The goal is to *reconstruct* P

Setup

- Measurement operator $\mathcal{A}$: $\mathcal{A}(\mathbf{X})_\ell = \begin{cases} \langle \mathbf{w}_{\alpha_\ell}, \mathbf{X} \rangle & \text{for } \ell \leq m \\ \langle \mathbf{w}_{(\ell-m,\ell-m)}, \mathbf{X} \rangle & \text{for } \ell > m \end{cases}$ $|\boldsymbol{\Omega}| = m$

$$\mathbf{w}_\alpha = \begin{cases} \mathbf{e}_i\mathbf{e}_i^\top + \mathbf{e}_j\mathbf{e}_j^\top - \mathbf{e}_i\mathbf{e}_j^\top - \mathbf{e}_j\mathbf{e}_i^\top, & \text{if } \alpha = (i,j) \in \mathbb{I}, \\ \frac{1}{2}(\mathbf{e}_i\mathbf{1}^\top + \mathbf{1}\mathbf{e}_i^\top), & \text{if } \alpha = (i,i) \text{ for some } i \in \{1,\ldots,n\}, \end{cases}$$

- Ground truth $\boldsymbol{X}^{\mathbf{0}} = \boldsymbol{P}^{\boldsymbol{T}}\boldsymbol{P}$

- Standard Coherence factor : ν [Tasissa and Lai]

- We solve the rank minimization problem defined by,

$$\min_{\mathbf{X} \in S_n} \operatorname{rank}(\mathbf{X})$$

Such that $\mathbf{X} \succeq 0$ and $\mathcal{A}(\mathbf{X}) = [\mathbf{D}_\Omega; \mathbf{0}]$

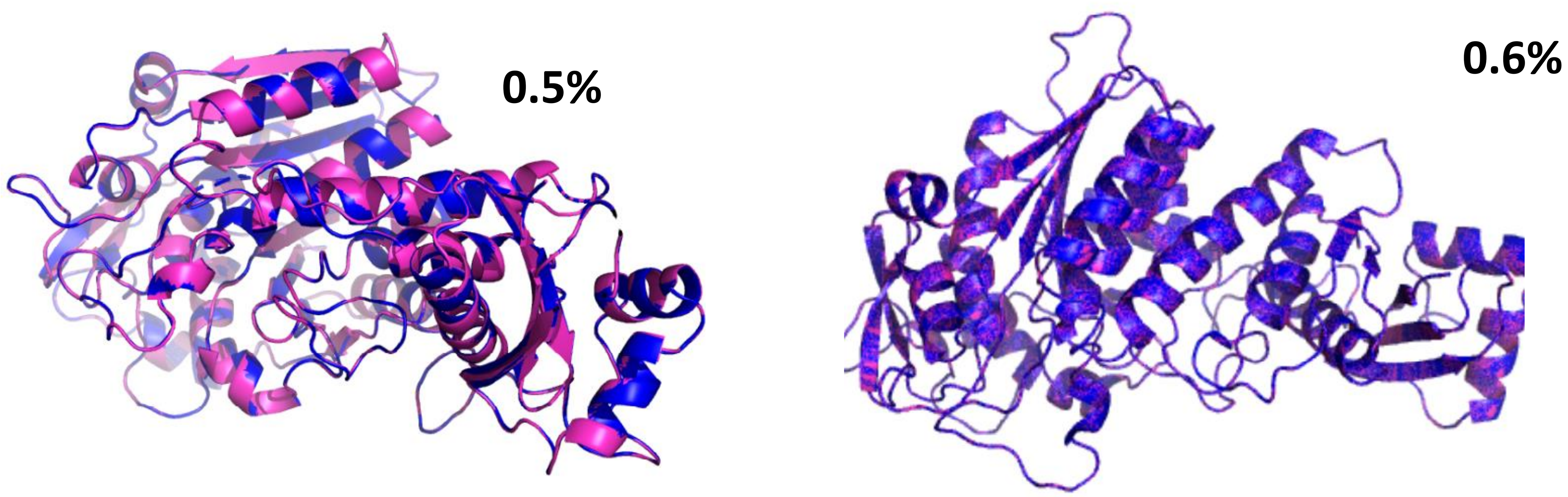
Limitations of Existing work

- Lack of computationally efficient algorithms for Nuclear Norm Minimization(NNM)
- Lack of Restricted Isometry Property (RIP) for Euclidean distance geometry problems

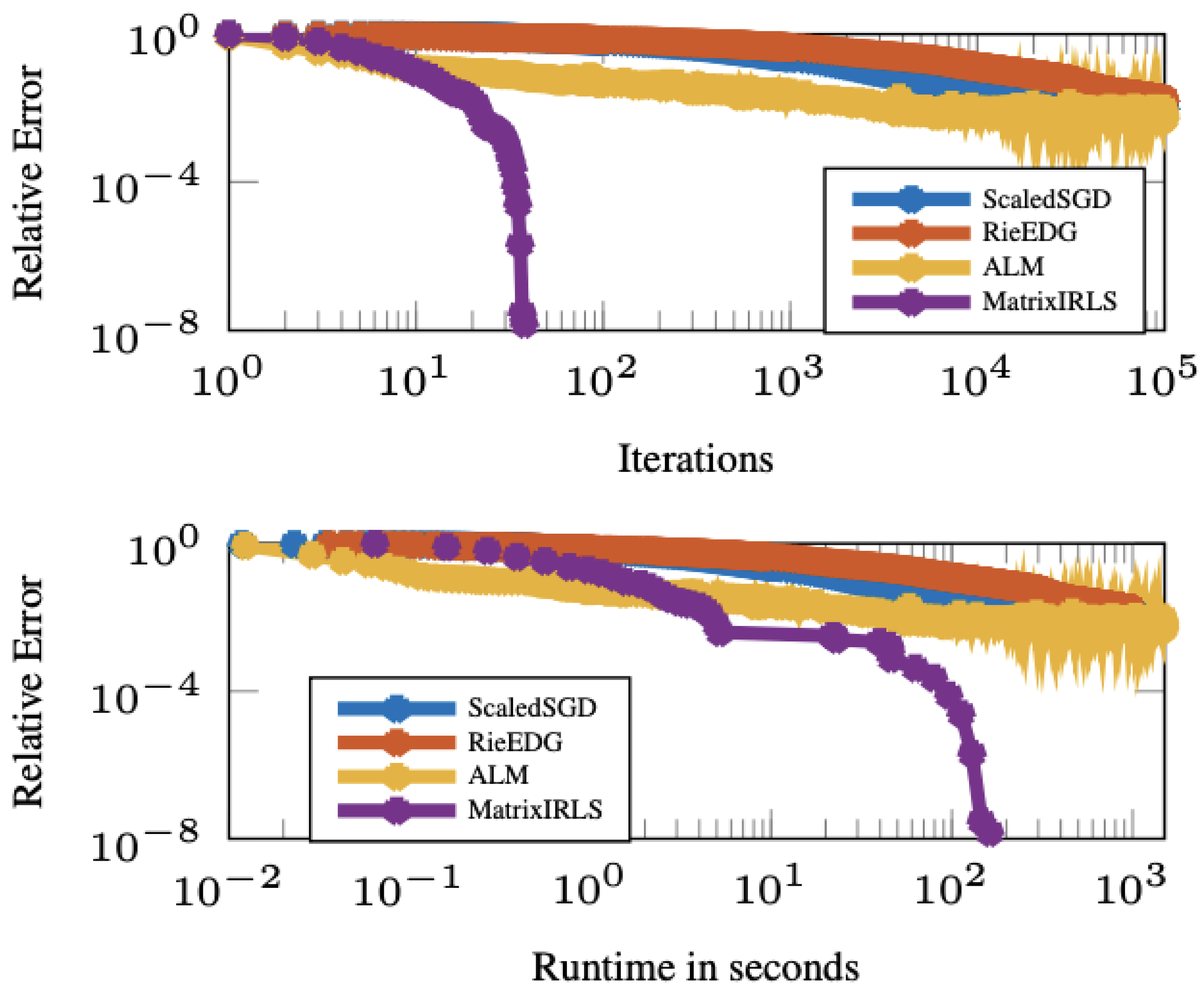
Our Contribution

- An algorithm based on Iteratively reweighted least squares framework (MatrixIRLS[1]).
  It implicitly minimizes smoothed log-det objectives by minimizing a quadratic model
- Local Convergence at optimal sample complexity
- Dual Basis formulation and establishing Restricted Isometry Property (RIP)

Experiments



Protein reconstruction by MatrixIRLS with 0.5% and 0.6% samples respectively



- MatrixIRLS shows reconstruction from fewer samples compared to other methods.
- MatrixIRLS is robust to ill-conditioned data
- Time to convergence for MatrixIRLS is significantly less than the other methods.

References

Reference:

[1] C. Kümmerle, C. Mayrink Verdun. A Scalable Second Order Method for Ill-Conditioned Matrix Completion from Few Samples, *ICML 2021*.

[2] C. Smith, H. Cai, A. Tasissa. Riemannian Optimization for Non-convex Euclidean Distance Geometry with Global Recovery Guarantees

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[4] A. Tasissa and R. Lai, "Exact reconstruction of Euclidean distance geometry problem using low-rank matrix completion," IEEE Trans. Inf. Theory, vol. 65, no. 5, pp. 3124–3144, 2019.

[5]Y. Li, X. Sun, "Sensor Network Localization via Riemannian Conjugate Gradient and Rank Reduction: An Extended Version", IEEE Transactions on Signal Processing