Minimum Entropy Coupling with Bottleneck

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Introduction

Let us consider the following general lossy compression setup:

$$X \xrightarrow{p_{T|X}} T \xrightarrow{q_{Y|T}} Y$$

Instead of expected sample-wise distortion $\mathbb{E}[d(X,Y)]$, we use the logarithmic-loss $H(X|Y)$.

Min. Ent. Coupling w/ Bottleneck

$$\begin{aligned} \max_{p_{T|X},\, q_{Y|T}} \quad & I(X;Y) \\ \text{s.t.} \quad & X \leftrightarrow T \leftrightarrow Y, \\ & H(T) \leq R, \\ & P(Y) = p_Y, \\ & P(X) = p_X \end{aligned}$$

Min. Ent. Coupling

$$\begin{aligned} \max_{p_{Y|X}} \quad & I(X;Y) \\ \text{s.t.} \quad & P(Y) = p_Y, \\ & P(X) = p_X \end{aligned}$$

Decomposition

Min. Ent. Coupling w/ Bottleneck (MEC-B)

$$X \xrightarrow{p_{T|X}} T \xrightarrow{q_{Y|T}} Y \qquad \max_{p_{T|X},\, q_{Y|T}} I(X;Y)$$

$P(X) = p_X \qquad H(T) \leq R \qquad P(Y) = p_Y$

Entropy-bounded Info. Max. (EBIM)

$$X \xrightarrow{p_{T|X}} T \qquad \max_{p_{T|X}} I(X;T)$$

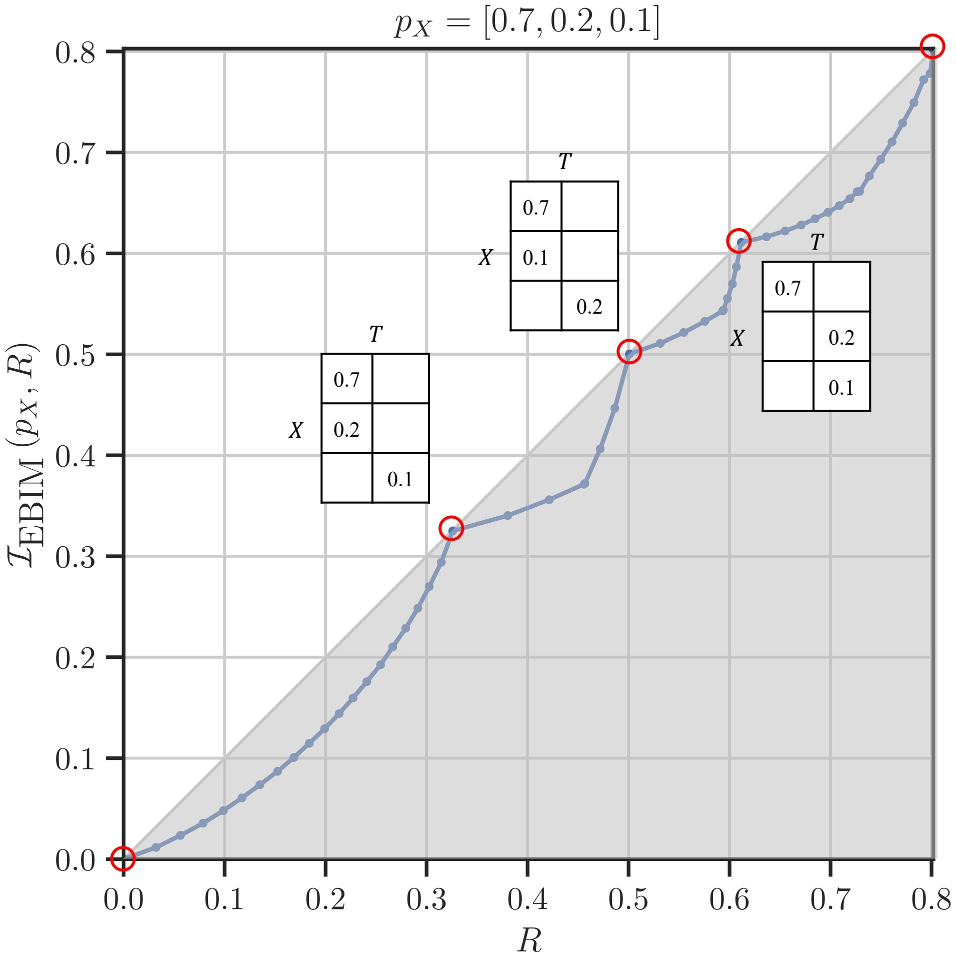
$P(X) = p_X \qquad H(T) \leq R$

Min. Ent. Coupling (MEC)

$$T \xrightarrow{q_{Y|T}} Y \qquad \max_{q_{Y|T}} I(Y;T)$$

$P(T) = p_T \qquad P(Y) = p_Y$

Entropy-Bounded Info. Max. (EBIM)



Entropy-Bounded Info. Max. (EBIM)

$$\mathcal{I}_{\text{EBIM}}(p_X, R) = \max_{p_{XT}} \quad I(X;T)$$
$$\text{s.t.} \quad H(T) \le R,$$
$$P(X) = p_X$$

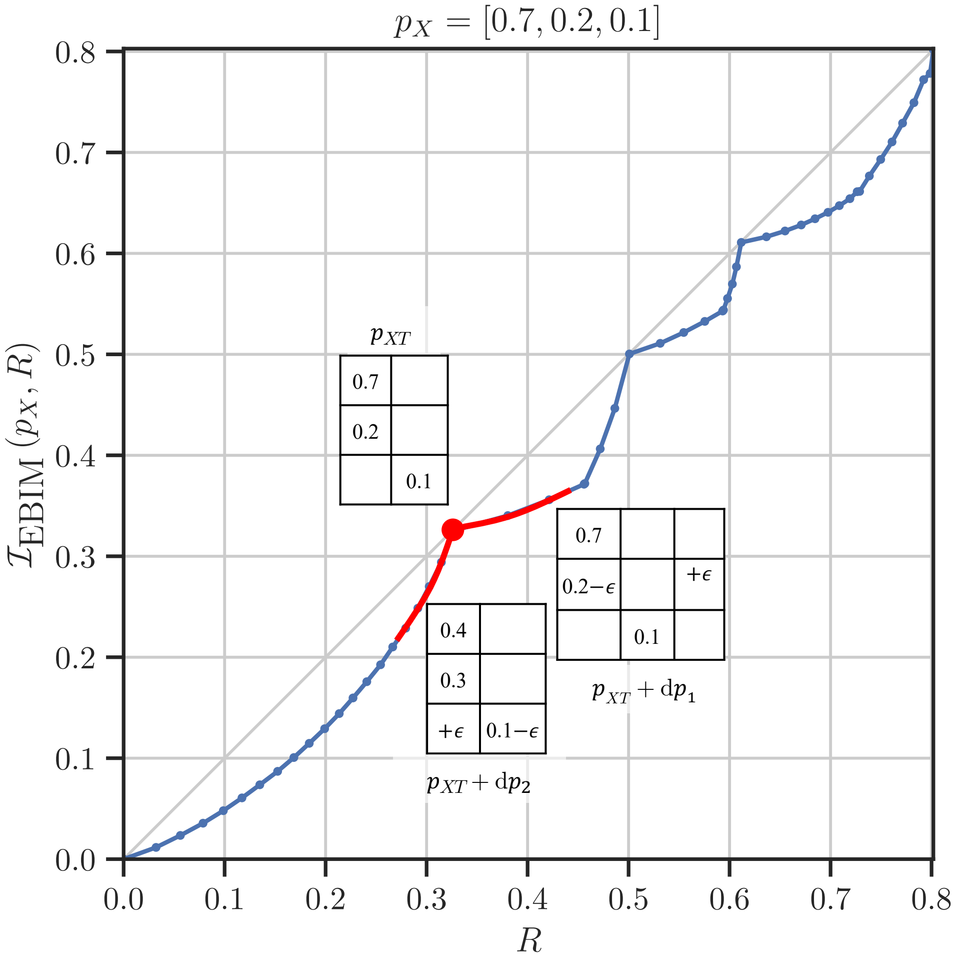
Upper bound

$$\mathcal{I}_{\text{EBIM}}(p_X, R) \le R$$

Theorem 1

$$\mathcal{I}_{\text{EBIM}}(p_X, R) = R$$
$$\iff \exists g : \mathcal{X} \to \mathcal{T} \text{ s.t. } H(g(X)) = R$$

Entropy-Bounded Info. Max. (EBIM)

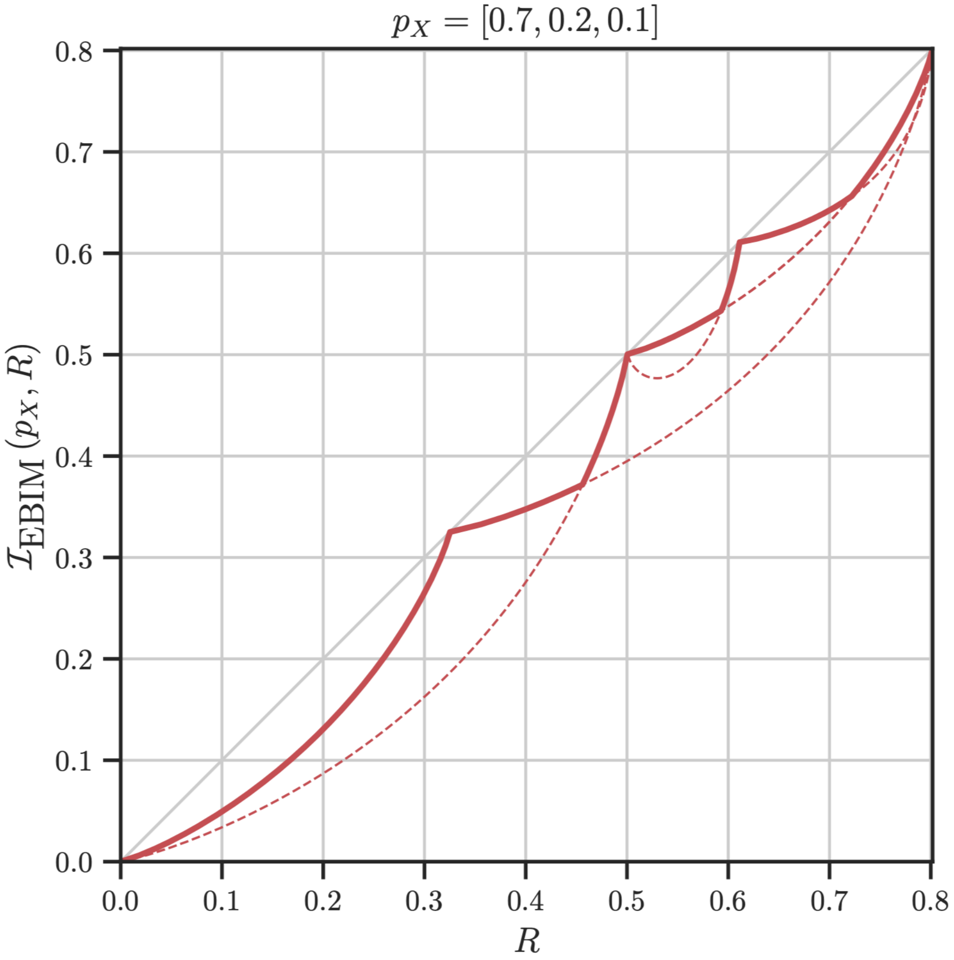


Theorem 3

Around a deterministic mapping $T = g(x)$, defines transformations resulting in optimal solutions for

1. $\mathcal{I}_{EBIM}(p_X, R_g + \epsilon)$
2. $\mathcal{I}_{EBIM}(p_X, R_g - \epsilon)$

Entropy-Bounded Info. Max. (EBIM)



Theorem 3

Around a deterministic mapping $T = g(x)$, defines transformations resulting in optimal solutions for

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Deterministic EBIM Solver

- The number of deterministic mappings in EBIM formulation is $O(n^n)$, where $n = |\mathcal{X}|$.
- Iterating over all deterministic mappings is not feasible.
- One should look for carefully constructed search algorithms to find such mappings with resulting $I(X;T)$ as close as possible to R.

Algorithm 5 Deterministic EBIM Solver

Input: p_X, R
Output: p_{XT}

$p_{XT} \leftarrow \text{diag}(p_X)$
for $i \leftarrow 1$ to $|p_X| - 1$ **do**
    $p_s^{(i)} \leftarrow$ Merge the two columns with the smallest sum in p_{XT}.
    $I_s^{(i)} \leftarrow$ Mutual Information imposed by $p_s^{(i)}$.
    $p_l^{(i)} \leftarrow$ Merge the two columns with the largest sum in p_{XT}.
    $I_l^{(i)} \leftarrow$ Mutual Information imposed by $p_l^{(i)}$.
    if $I_s^{(i)} \leq R$ **then**
        return $p_s^{(i)}$
    else if $I_l^{(i)} \leq R < I_s^{(i)}$ **then**
        return $p_l^{(i)}$
    else
        $p_{XT} \leftarrow p_l^{(i)}$

Theorem 2

If the output of Algorithm 5 yields mutual information $\widehat{I}$, then:

$$\mathcal{I}_{EBIM}(p_X, R) - \widehat{I} \leq h(p_2),$$

where $h(\cdot)$ is the binary entropy function, and p_2 denotes the second largest element of p_X.

Core Contributions

Minimum Entropy Coupling with Bottleneck

- We presents a lossy compression framework under logarithmic loss, that extends Min. Entropy Coupling (MEC) with a bottleneck.
- We then propose Entropy-Bounded Information Maximization (EBIM) formulation for the encoder, extensively characterize the structure of its optimal solution, and provide efficient approximate solutions with guaranteed performance.
- We also illustrate the practical application of MEC-B through experiments in Markov Coding Games under rate limits.