Barely Random Algorithms and Collective Metrical Task Systems

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5 mins

Warmup: Zero-sum Games

M be a **cost** matrix in 2-player game

Cost	*Rock*	*Paper*	*Scis.*
Rock	0	1	-1
Paper	-1	0	1
Scis.	1	-1	0

	« Pure »	**« Barely Random » (k=2)**	**« Mixed »**
Strategy x =	$(\mathbf{1}, \mathbf{0}, \mathbf{0})$	$\left(\frac{\mathbf{1}}{\mathbf{2}}, \frac{\mathbf{1}}{\mathbf{2}}, \mathbf{0}\right)$	$\left(\frac{\mathbf{1}}{\mathbf{3}}, \frac{\mathbf{1}}{\mathbf{3}}, \frac{\mathbf{1}}{\mathbf{3}}\right)$
Cost	1	0.5	0

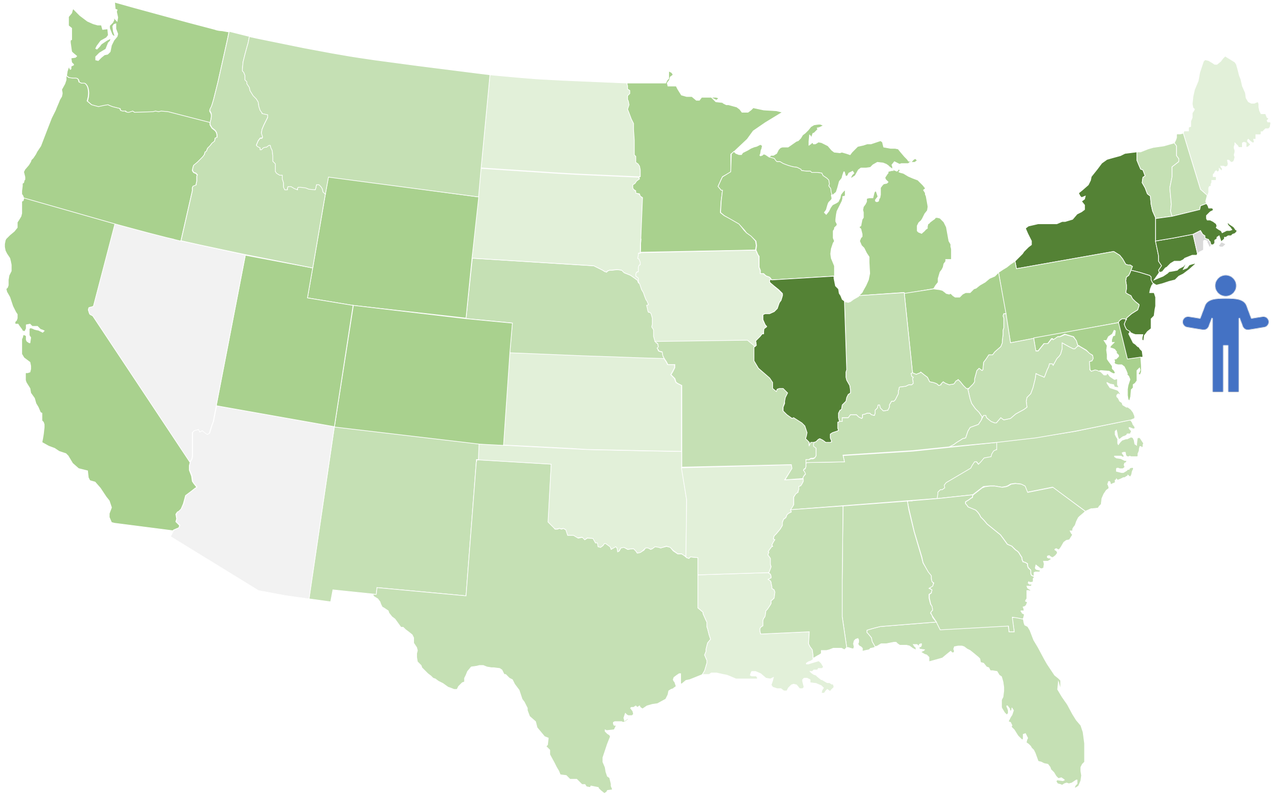
Definition. A $k \in \mathbb{N}$ −barely **random strategy** $\boldsymbol{x} \in \mathcal{P}_{\boldsymbol{k}}$ is one which

- Takes value in $\{0, \frac{1}{k}, \frac{2}{k}, \dots, 1\}$ — *(k-barely fractional)*
- Can be sampled with a random seed $S \sim \mathcal{U}(\{1, \dots, k\})$ — *(k-barely random)*
- Represents the distribution of k players playing *collectively* — *(k-collective)*

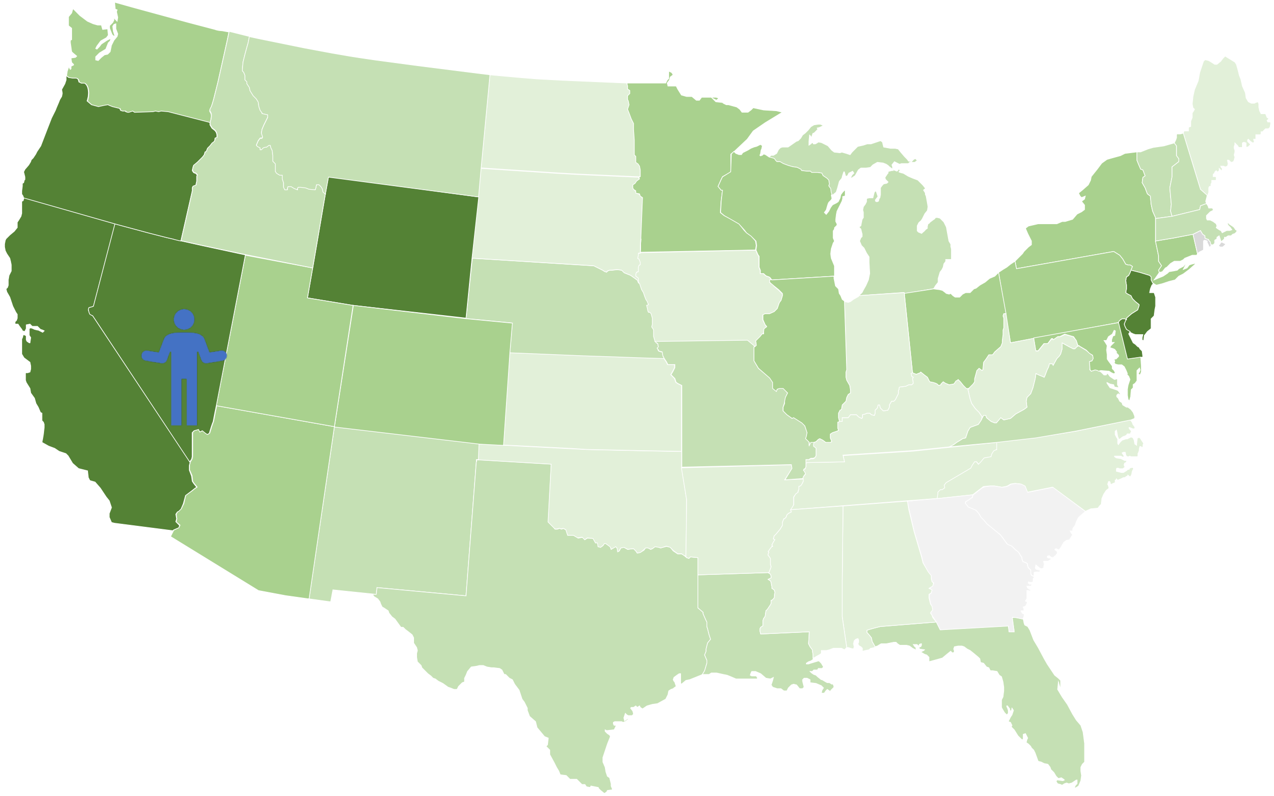
How does the payoff/cost scales with « barelyness » $k \in \mathbb{N}$?

In the context of real-time games (online algorithms)

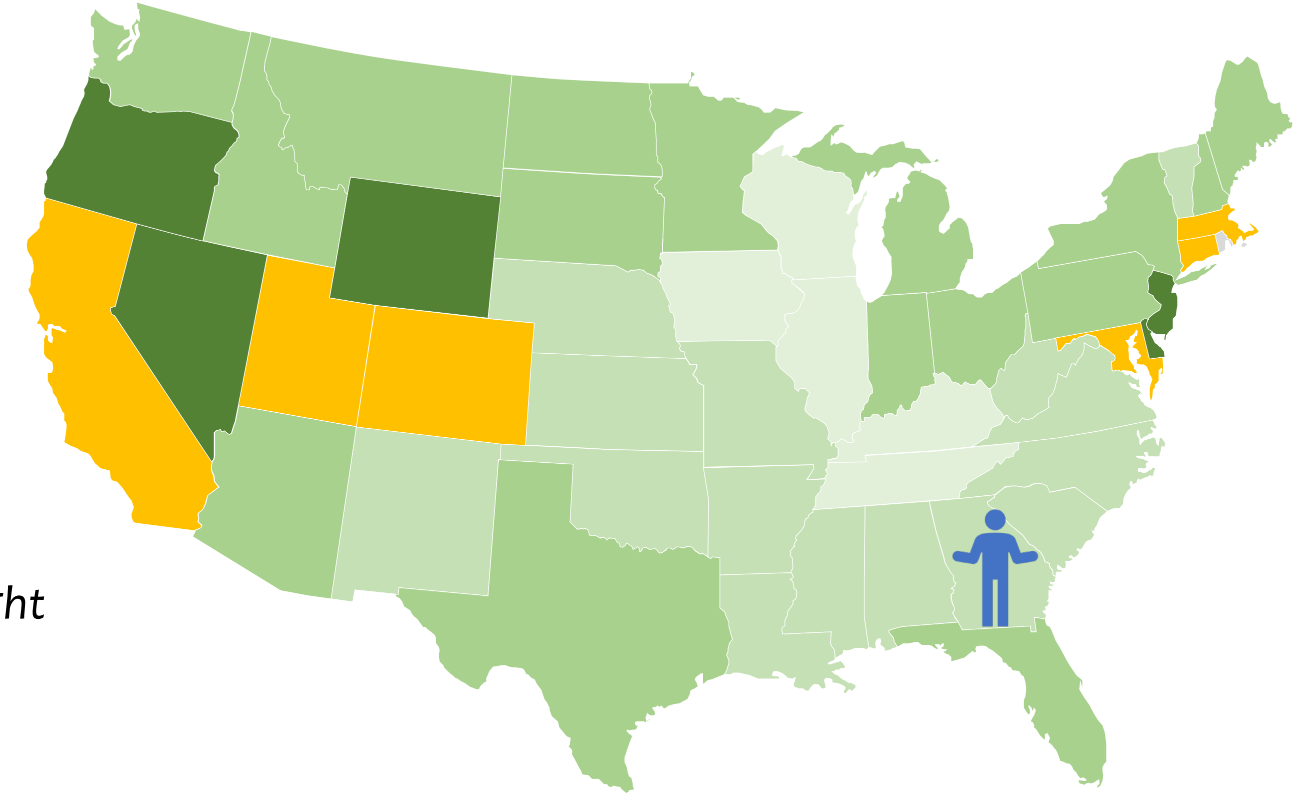
Price of Rent in 1950



Price of Rent in 1980



Price of Rent in 2000



$$Comp.\,Ratio = \frac{\textbf{Cost}}{\textbf{OPT}}$$

= *cost due to lack of hindsight*

Metrical Task Systems

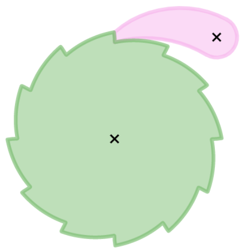
When should I move? (between jobs, cities, power saving modes …)

- Metric space $(\mathcal{X}, d)$, with $|\mathcal{X}| = n$ number of positions.
- Input $\mathbf{c}(\cdot)$: Cost vector $\mathbf{c}(\mathbf{1}), \mathbf{c}(\mathbf{2}), \ldots \in \mathbb{R}_+^{\mathcal{X}}$
- Output $x(\cdot)$: Agent's position $x(1), x(2), \ldots \in \mathcal{X}$ or $\boldsymbol{k} \in \mathbb{N}$ (barely random) positions $\in \mathcal{P}_k(\mathcal{X})$
  - With past data $\mathbf{c}(\leq \boldsymbol{t})$
- **Cost** = Cost vector + Movement, i.e., $\mathrm{Cost}\,(x(\cdot), \mathbf{c}(\cdot)) = \sum_t c_{x(t)}(t) + d(x(t), x(t+1))$
- **OPT** = Best with hindsight, i.e. $\mathbf{OPT} = \inf_{x(\cdot)} \mathrm{Cost}\,(x(\cdot), \mathbf{c}(\cdot))$

Main result

	Deterministic : $k = 1$	« Barely Random » $k \in \mathbb{N}$	Randomized: $k = \infty$
Comp. Ratio	$\mathbf{2n - 1}$ [BLS]	$\boldsymbol{\Theta}(\mathbf{log}^2\, \boldsymbol{n})$ if $\boldsymbol{k} \geq \boldsymbol{n}^2$ [This work]	$\boldsymbol{\Theta}(\mathbf{log}^2\, \boldsymbol{n})$ [Bubeck et al. 2019-22]

Techniques:



physical device with *hysteresis:* *ratchet* (*green*)+ pawl (*purple*)

- **Idea:**
  - Transform a fully fractional strategy $\boldsymbol{y}(t) \in \mathcal{P}(X)$
  - In a k – barely fractional strategy $\boldsymbol{x}(t) \in \mathcal{P}_k(X)$

The issue (e.g. for $k = 2$)

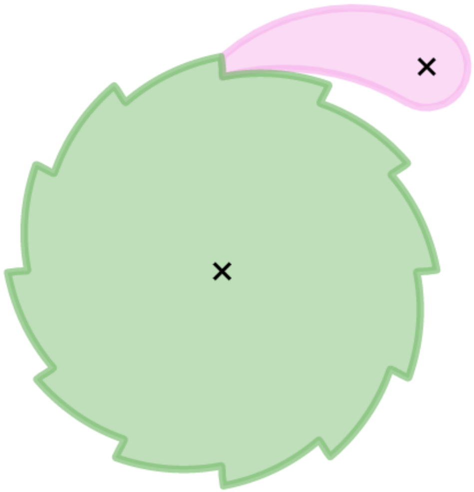
$\boldsymbol{y}(t)$	$\boldsymbol{x}(t)$
$(0.76, 0.24)$	$(1, 0)$
$(0.74, 0.25)$	$(0.5, 0.5)$

- **Naïve transform :**

$$x(t) = \arg\min_{x \in \mathcal{P}_k(\mathcal{X})} \mathrm{OT}(x, y(t))$$

- **Good transform :** *(Hysteresis!) or (« L1 Wasserstein Prox »)*

$$x(t) = \arg\min_{x \in \mathcal{P}_k(\mathcal{X})} \mathrm{OT}(x, y(t)) + \mathrm{OT}(x(t-1), x).$$



Thank you!

And think of Hysteresis to build Robust systems in face of Uncertainty