“Unsupervised Discovery of Formulas for Mathematical Constants”

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*Equal contribution

AI for Math

- **Decades of previous efforts, mostly on Automated Theorem Provers (ATP).**
  - Fajtlowicz (1988), Petkovsek et al. (1996), Wolfram et al. (2002), Buchberger et al. (2006), Bailey et al. (2007), Davies et al. (2021), Fawzi et al. (2022), Trinh et al. (2024).

- **Historically number theory and specifically discovering formulas for mathematical constants proved challenging for AI methods.**

Polynomial Continued Fractions

$$a_0 + \cfrac{b_1}{a_1 + \cfrac{b_2}{\ddots + \cfrac{b_n}{a_n}}} = \frac{p_n}{q_n} \to L$$

$$a_n = A_2 n^2 + A_1 n + A_0$$
$$b_n = B_2 n^2 + B_1 n + B_0$$

$$1 + \cfrac{1}{3 + \cfrac{4}{\ddots + \cfrac{n^2}{2n + 1 + \ddots}}} = \frac{4}{\pi} - 1$$

$$2 + \cfrac{1}{4 + \cfrac{4}{\ddots + \cfrac{n^2}{2n + 2 + \ddots}}} = 0.218599007\ldots$$

The Numerical Limit is a Bad Metric

$$1 + \cfrac{-1}{3 + \cfrac{-1}{\ddots + \cfrac{-1}{2n + 1 + \ddots}}} = \cot(1) = 0.64209261\ldots$$

$$\cfrac{5}{8 + \cfrac{-5}{\ddots + \cfrac{-5n^2 + 5n + 5}{4n^2 + 4n + \ddots}}} = 0.642097789\ldots$$

$$2 + \cfrac{-3}{2 + \cfrac{1}{\ddots + \cfrac{n^2 + n - 5}{n^2 - n + 2 + \ddots}}} = 0.642092687\ldots$$

Dynamical Metrics

$$\underbrace{2.71828\cdots}_{e} = 3 - \cfrac{1}{4 - \cfrac{2}{5 - \cfrac{3}{6 - \cfrac{4}{7 - \cfrac{5}{8 - \cdots}}}}}$$

$$\underbrace{3.50387\cdots}_{\cfrac{2}{\frac{\pi}{2} - 1}} = 3 + \cfrac{3}{5 + \cfrac{8}{7 + \cfrac{15}{9 + \cfrac{24}{11 + \cfrac{35}{13 + \cfrac{48}{\cdots}}}}}}$$

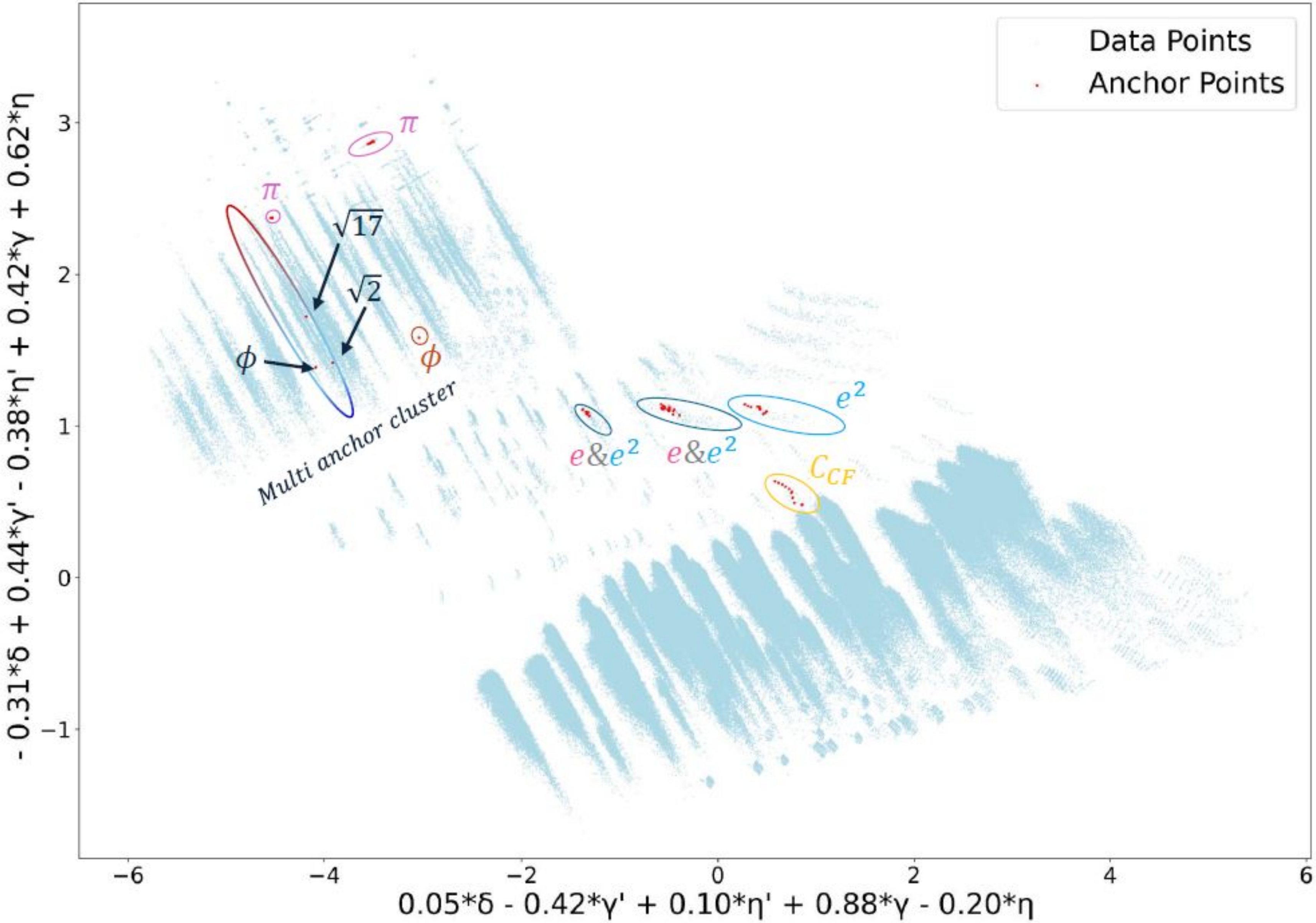
Dynamical Metrics

$$\epsilon(n) := \left|\frac{p_n}{q_n} - L\right| \quad \longrightarrow \quad \epsilon(n) \sim n!^{\eta} \cdot e^{\gamma n} \cdot n^{\beta}$$

$$\delta_n = \frac{-\log\left|L - \dfrac{p_n}{q_n}\right|}{\log|\tilde{q}_n|} - 1 \quad , \quad \tilde{q}_n = \frac{q_n}{\gcd(p_n, q_n)}$$

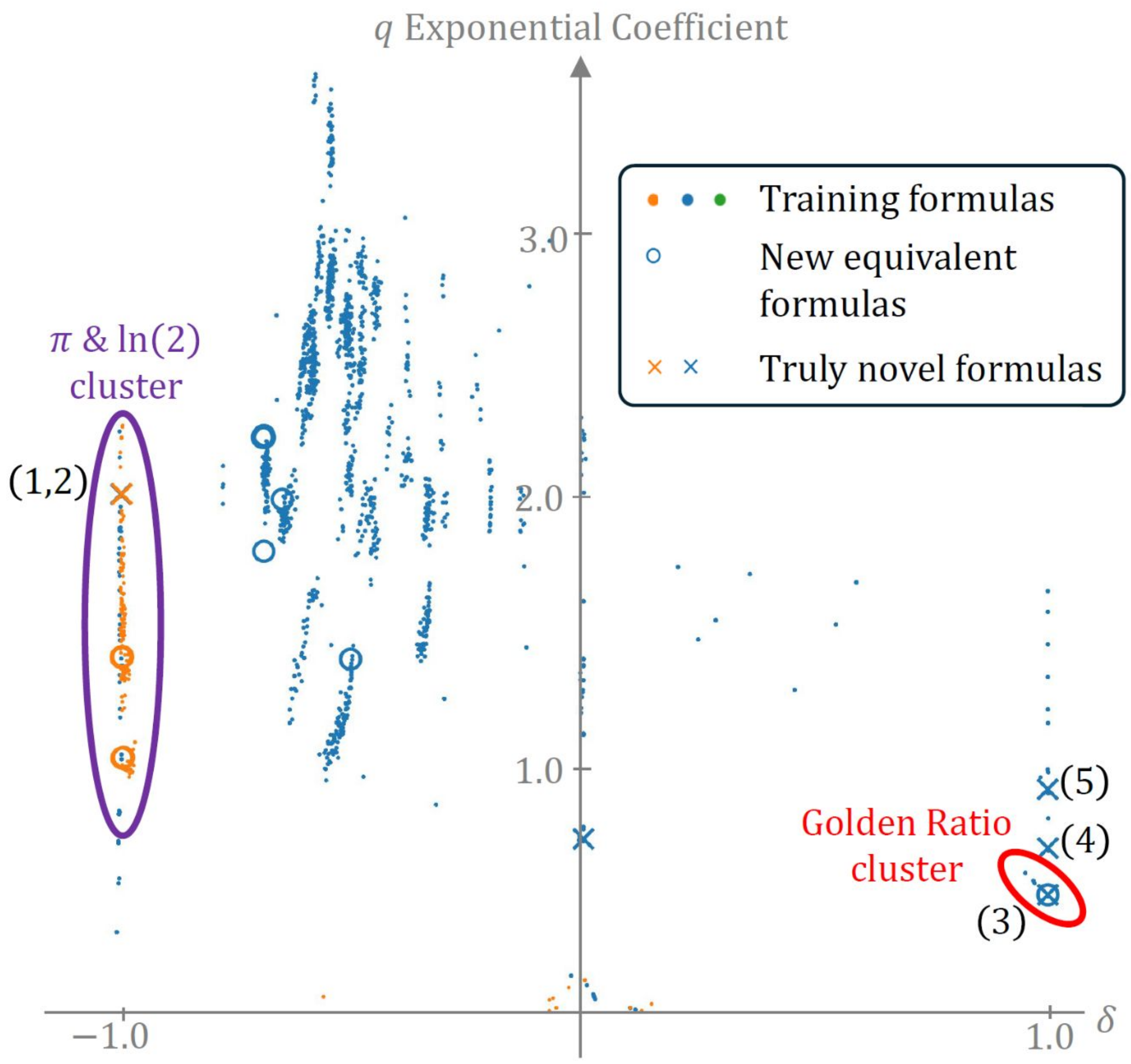
$$\tilde{q}_n \sim n!^{\eta'} \cdot e^{\gamma' n} \cdot n^{\beta'}$$

Clusters



Data Points
Anchor Points
π
π
√17
√2
ϕ
ϕ
Multi anchor cluster
e&e²
e&e²
e²
C_{CF}
3
2
1
0
−1
−6
−4
−2
0
2
4
6
- 0.31*δ + 0.44*γ' - 0.38*η' + 0.42*γ + 0.62*η
0.05*δ - 0.42*γ' + 0.10*η' + 0.88*γ - 0.20*η

Classifying Unseen Structures



Novel π formulas

(1) $\frac{6}{\pi^2} = 1 + \cfrac{-1}{5+\cfrac{-16}{13+\cfrac{-n^4}{2n^2+n+1+\ddots}}}$

(2) $\frac{12}{\pi^2} = 1 + \cfrac{1}{3+\cfrac{16}{5+\cfrac{n^4}{2n+1\ddots}}}$

Novel Golden Ratio formula

(3) $\phi = 1 + \cfrac{3}{3+\cfrac{21}{7+\cfrac{n^4+n^2+1}{n^2+n+1+\ddots}}}$

Novel square root formulas

(4) $1+\sqrt{3} = 2 + \cfrac{6}{6+\cfrac{42}{14+\cfrac{2n^4+2n^2+2}{2n^2+2n+2+\ddots}}}$

(5) $1+\sqrt{2} = 2 + \cfrac{3}{6+\cfrac{21}{14+\cfrac{n^4+n^2+1}{2n^2+2n+2+\ddots}}}$

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Raayoni et al., **Nature** 590, 67–73 (2021)

Razon et al., **ICML** 202, 28809-28842 (2023)

David et al., **arXiv** 2303.09318 (2023)

Elimelech et al., **PNAS** (2024)


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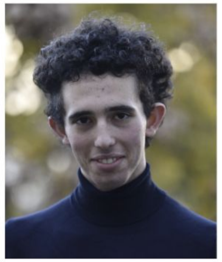
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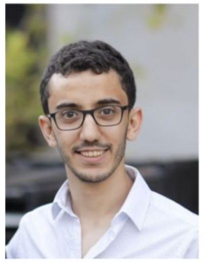
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