Almost Free: Self-concordance in Natural Exponential Families and an Application to Bandits

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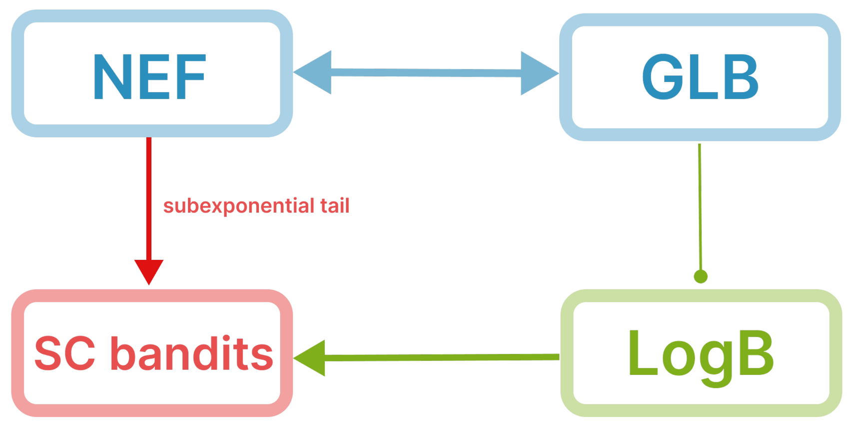
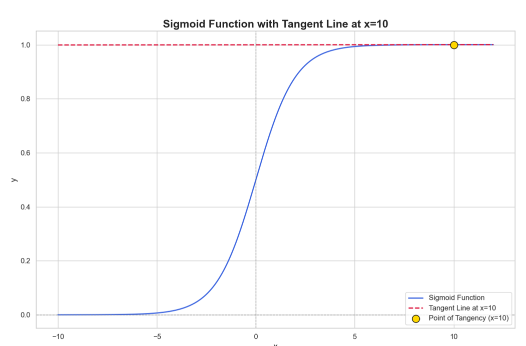
Summary

NEF with subexponential tail enjoys a good statistical and optimization property, self-concordance (SC). Applications include bandits, supervised learning, etc...

Overview



(a) GLB: $\mathbb{E}[r_x] = \mu(x^\top \theta_\star)$



(b) $\kappa \approx \exp(\|\theta_\star\|)$

(a) With unsophisticated analysis: $R(T) \approx \kappa\sqrt{T}$

(b) With SC on logistic bandits: $R(T) \approx \sqrt{T/\kappa} + \kappa$

(c) $\kappa = \frac{1}{\inf_{x\in\mathcal{X}} \dot\mu(x^\top\theta_\star)}$

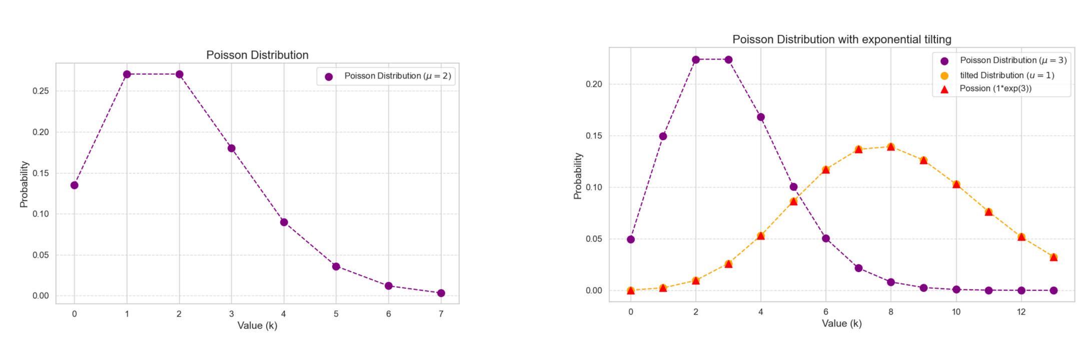
Background: NEF and GLB

NEF:

(a) Fix a base distribution Q over the reals

(b) $Q_u(dy) = \frac{\exp(uy)Q(dy)}{M_Q(u)}$ with u such that $M_Q(u) := \int_{\mathbb{R}} \exp(uy)Q(dy) < \infty$

(c) $\mu(u) = \int yQ_u(dy)$, $\frac{d^n}{du^n}\mu(u) = \int (y-\mu(u))^{n+1} Q_u(dy)$ for $n \in \{1, 2\}$



GLB: $u_x = x^\top \theta_\star$

Self-concordance Property

For a thrice-differentiable real-valued function $\mu : D \to \mathbb{R}$, there exists a function $\Gamma : D \to \mathbb{R}_{\geq 0}$ such that

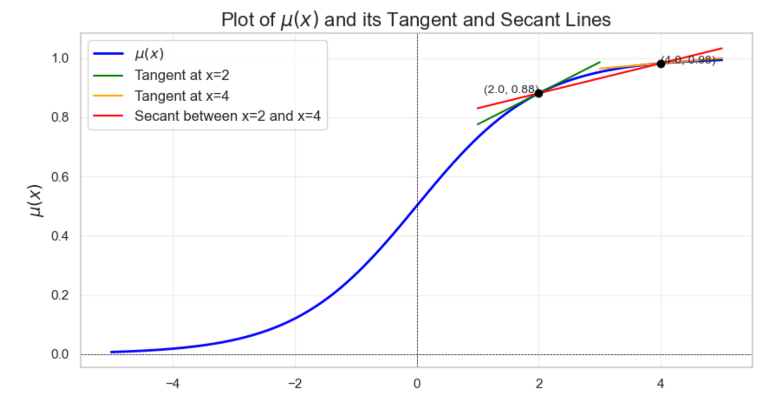
$$|\ddot\mu(u)| \leq \Gamma(u)\dot\mu(u)$$

for all $u \in D$. It allows us to control the ratio between the derivatives:

$$\dot\mu(u')\exp(-R|u-u'|) \leq \dot\mu(u) \leq \dot\mu(u')\exp(R|u-u'|)$$

$$\frac{\dot\mu(u)}{1+R|u-u'|} \leq \int_0^1 \dot\mu(u+t(u'-u))dt = \frac{\mu(u)-\mu(u')}{u-u'}.$$

where $R = \sup_{x\in D}\Gamma(x)$.



This gives a more sophisticated way of constructing the confidence set using local curvature of $\mu(\cdot)$ which avoids paying κ in the first order term of the regret bound.

Research Question

The analysis of LogBs suggests that we are able to get rid of κ in the first-order term for GLBs whose corresponding NEF has a self-concordant mean function.

Question: How large is the class of NEF with self-concordant mean functions?

Short Answer: Upper Bounds

Main Theorem:
If an NEF has a subexponential base distribution, then there exists a polynomial function $\Gamma(\cdot)$ such that the mean function $\mu(\cdot)$ is Γself-concordant.

Upper Bounds

Subexponential base Q: NEF $(Q_u)_{u\in[-c;c]}$ is self-concordant with function $\Gamma(u)$ s.t.:

$$\Gamma(u) = O\left(\frac{1}{(c-|u|)^2}\right).$$

Subgaussian base Q: NEF $(Q_u)_{u\in\mathbb{R}}$ is self-concordant with function $\Gamma(u)$ s.t.:

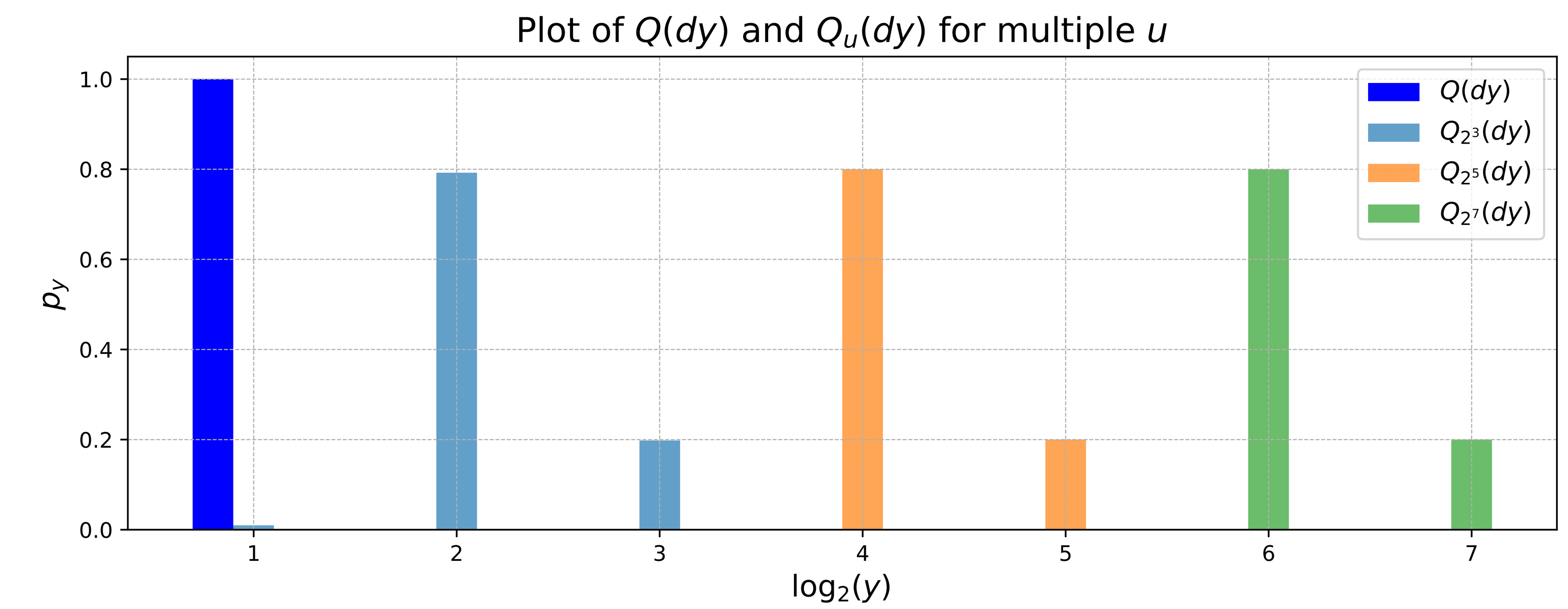
$$\Gamma(u) = O(|u|).$$

Lower Bounds

Subexponential base Q: Taking Q to be Exp(c), we have: $\Gamma(u) \geq \left(\frac{1}{c-u}\right)$.

Subgaussian base Q: UB is order tight for a certain base distribution Q:

- If $X \sim M\mathcal{B}(p)$, with $0 < p < \frac{1}{2}$, then $\frac{\mathbb{E}[(X-\mathbb{E}[X])^3]}{\mathbb{E}[(X-\mathbb{E}[X])^2]} = O(M)$.
- Q constructed so that $Q_{2^i} \approx$ cst $+ 2^i\mathcal{B}(p)$.



Open Questions

- Can we find tight bounds for subexponential NEF?
- Can we extend beyond NEF?

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