Alternating Gradient Flows: A Theory of Feature Learning in Two-layer Neural Networks

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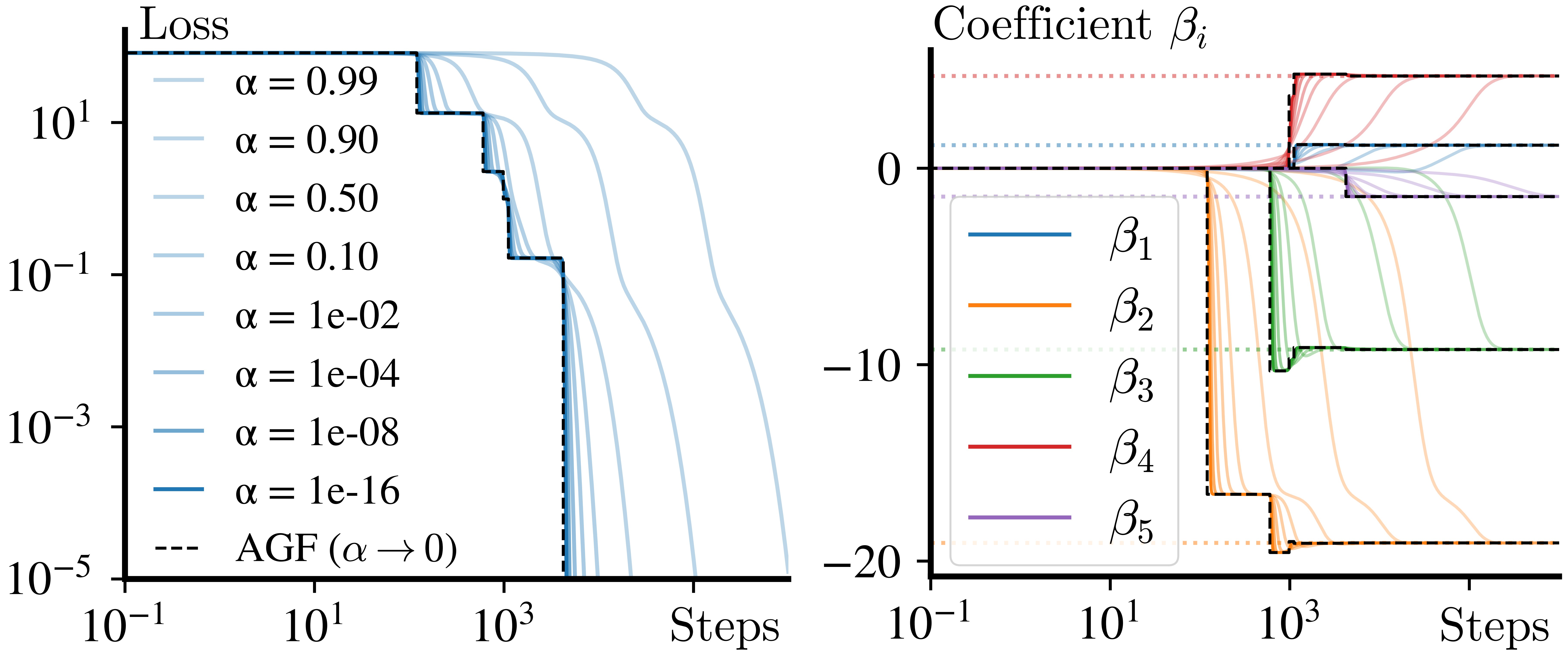
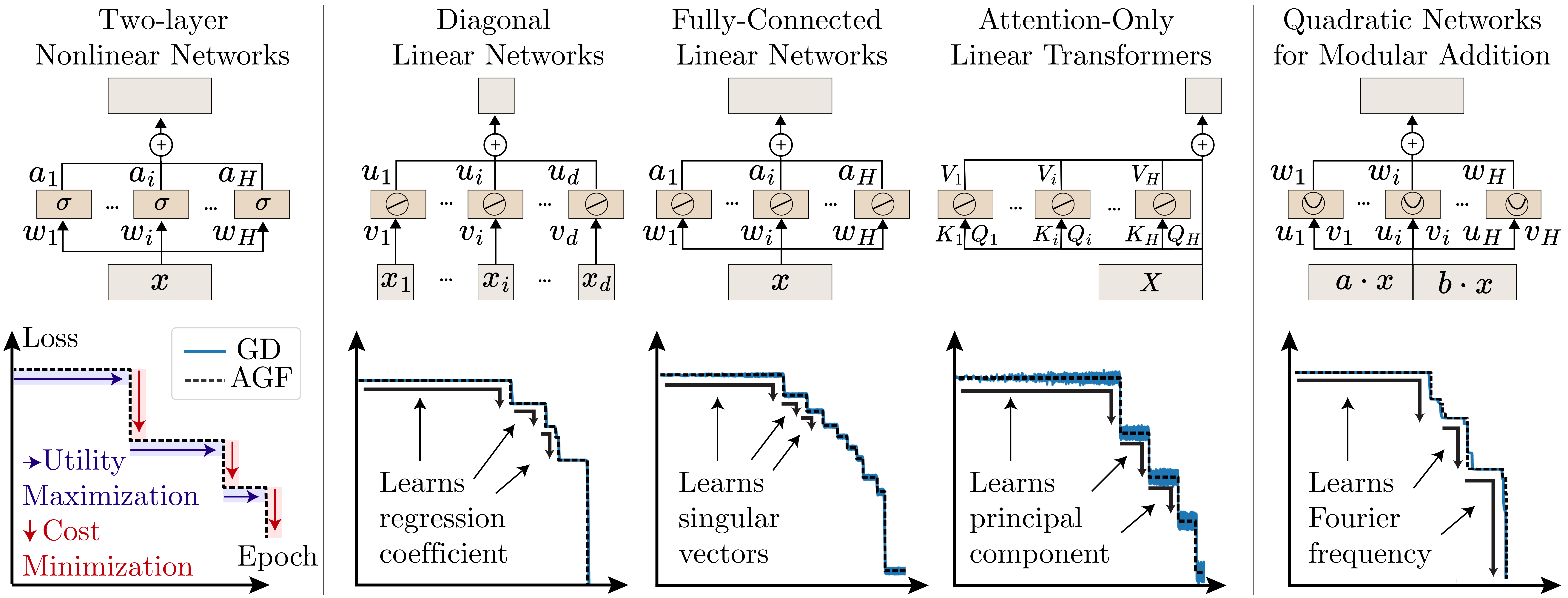
Kunin et al. (2025)

TL;DR: We introduce Alternating Gradient Flows, a framework modeling feature learning in <u>two-layer</u> networks with <u>vanishing initialization</u> as alternating utility maximization and cost minimization steps



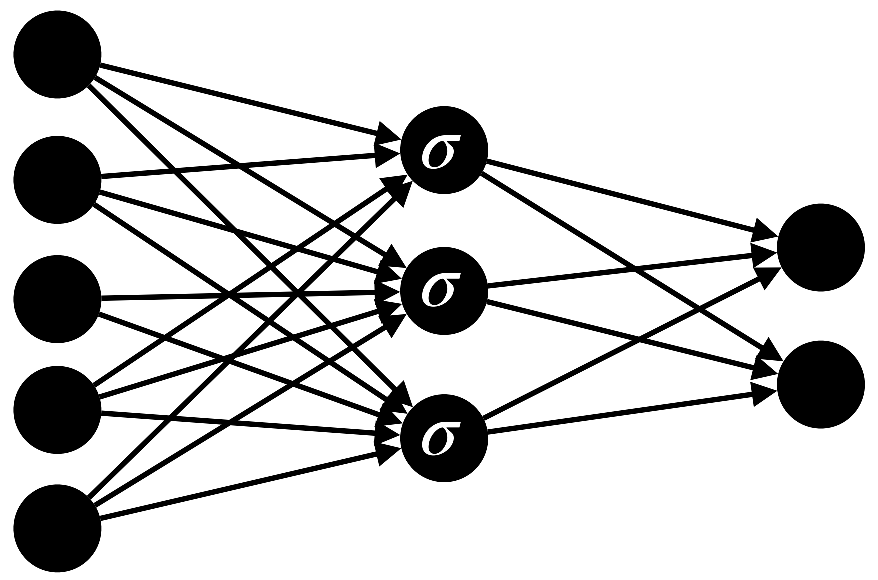
AGF — a step towards a broader understanding of feature learning



Part 1: Deriving a Two-step Algorithm (AGF) that Approximates Gradient Flow

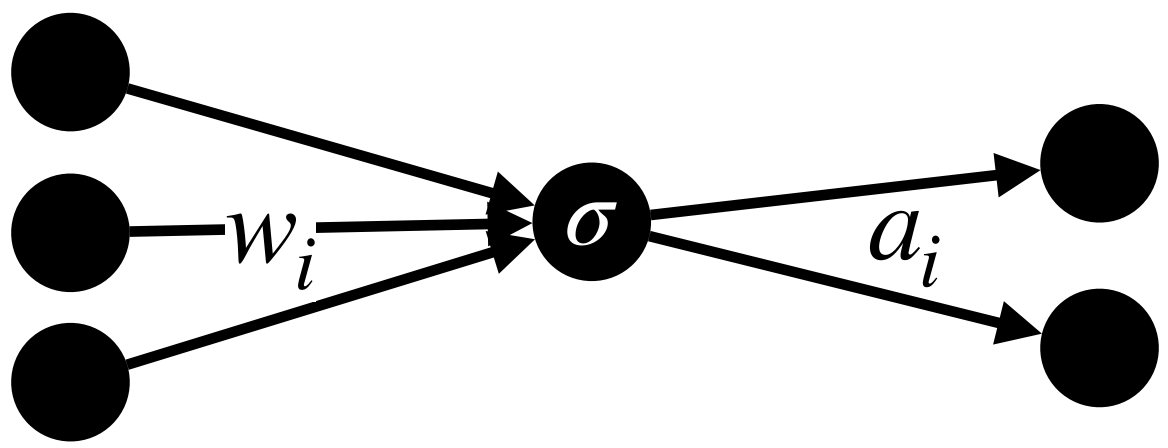
Two-layer network

$$f(x;\Theta) = \sum_{i=1}^{H} f_i(x;\theta_i)$$



"Neuron"

$$f_i(x;\theta_i) = a_i\sigma(w_i^\intercal x)$$

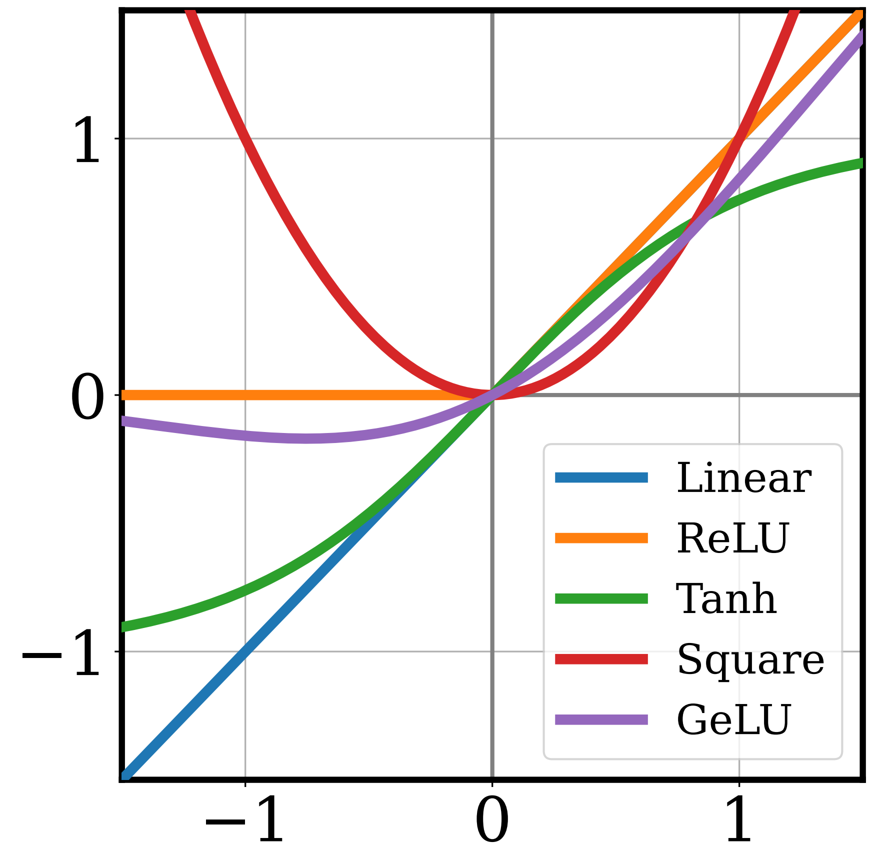


Parameters initialized

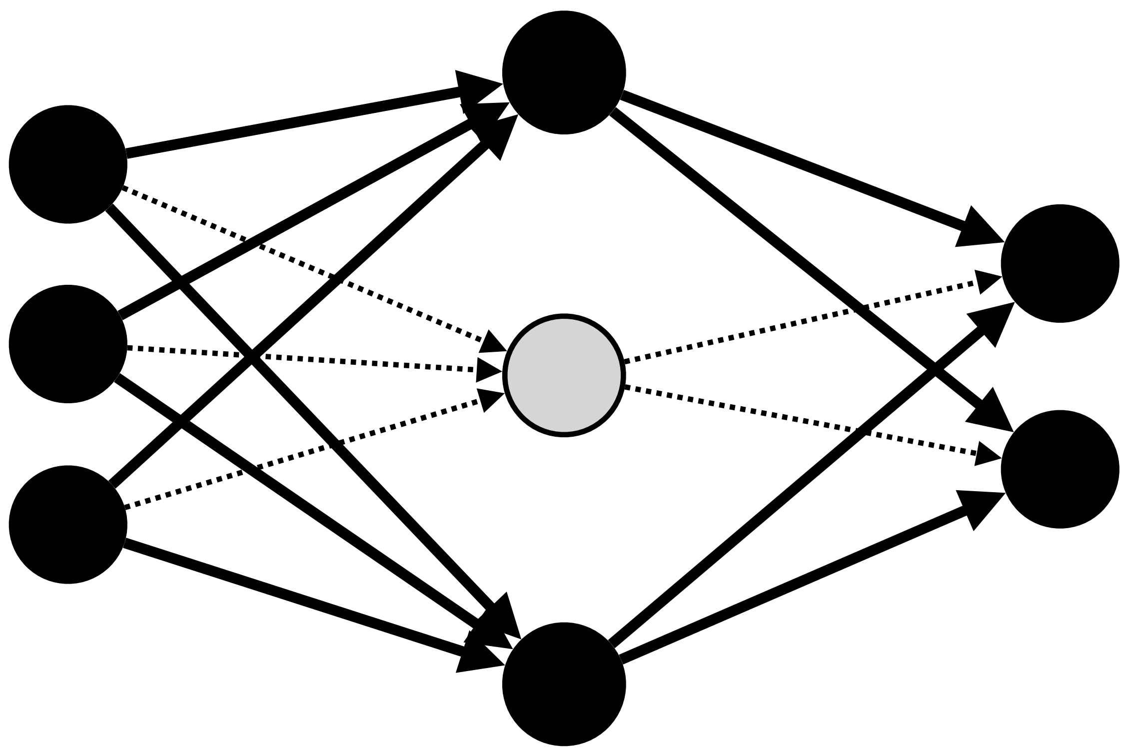
$\Theta_0 \sim \mathcal{N}(0,\alpha)$ with $\alpha \ll 1$

Origin-passing

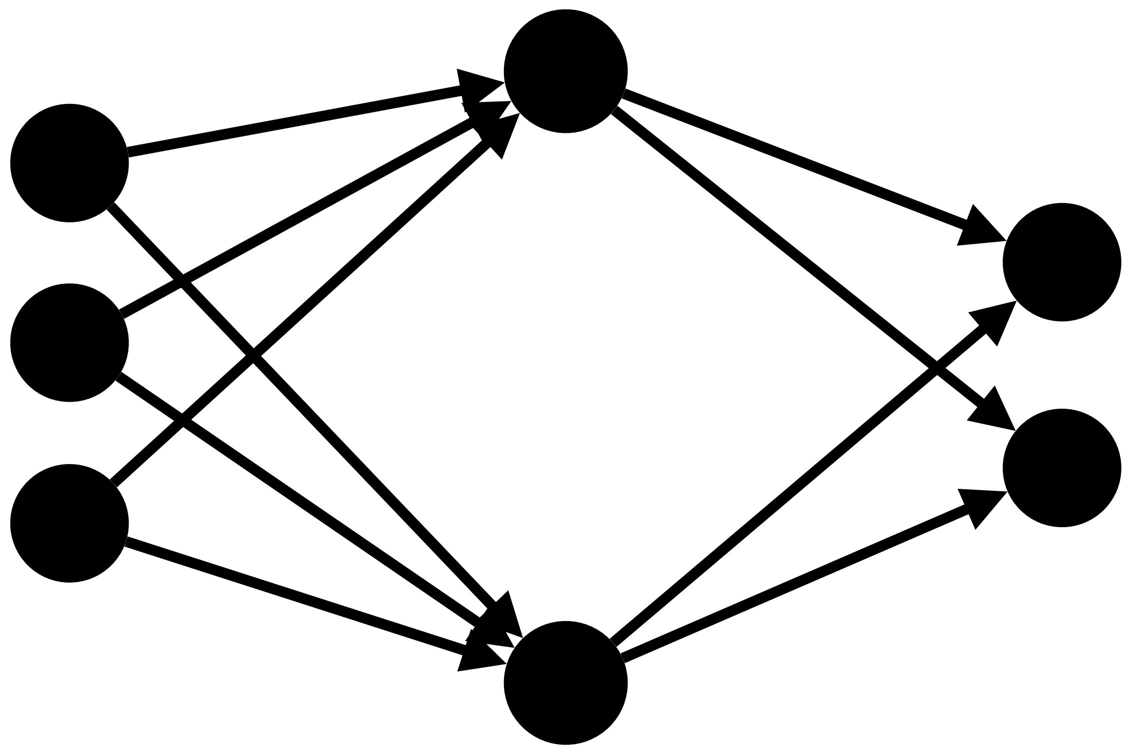
$$\sigma(0) = 0$$



Observation 1: If $\theta_i = 0$, then it will stay zero throughout training (i.e. $\nabla_{\theta_i}\mathscr{L}(\Theta) = 0$)



Observation 2: Neurons are only "aware" of other neurons through the residual $r(x;\Theta) = y(x) - f(x;\theta)$



Alternating Gradient Flows (AGF)

Initialize: $\mathscr{D} = [H], \mathscr{A} = \{\}, r(x) = y$

Utility Maximization:

For each $i \in \mathscr{D}$, Gradient ascent $\mathscr{U}_i(\theta_i, r)$ subject to $\|\theta_i\| = 1.$

Transition neuron with smallest jump time τ_i to active set.

Cost Minimization:

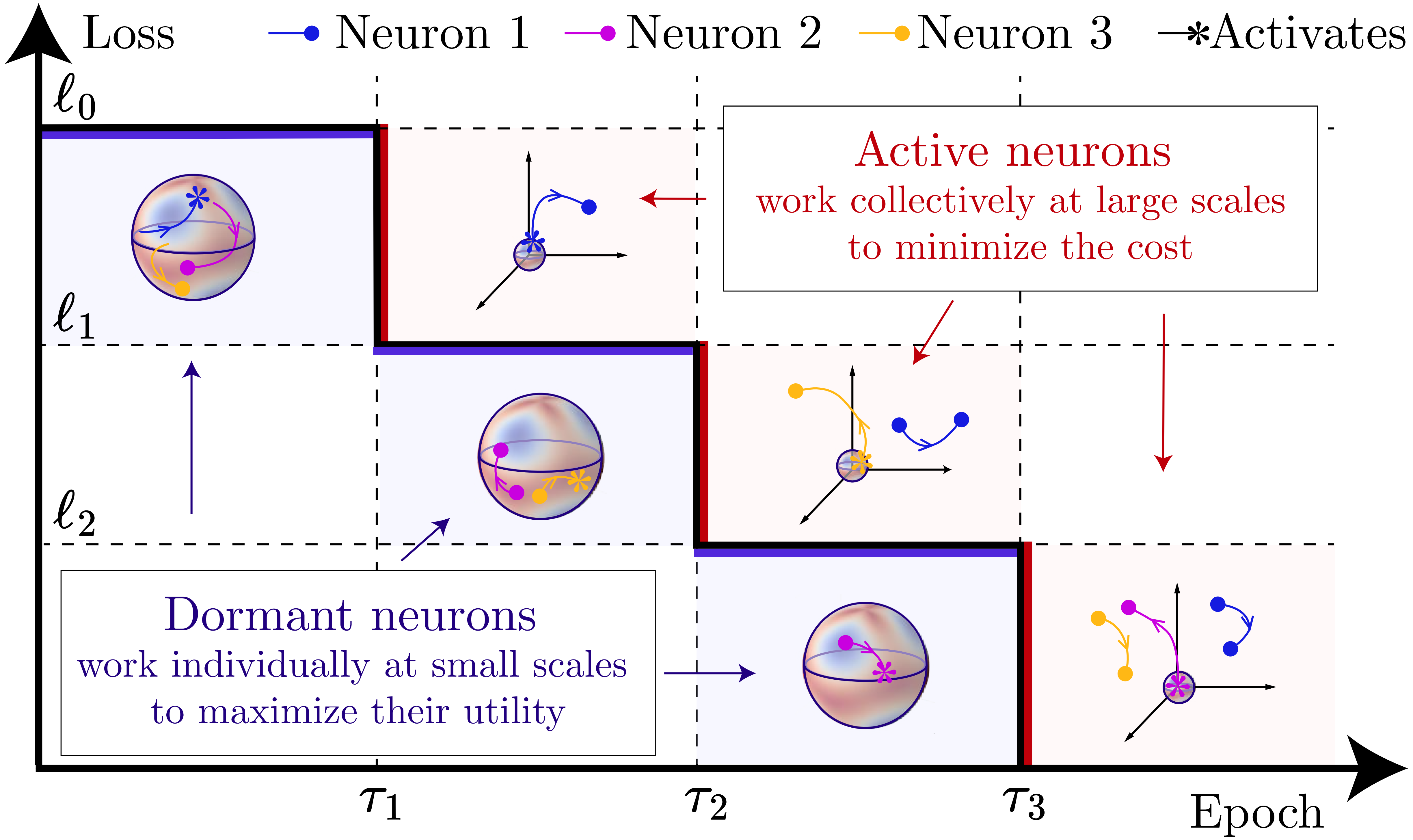
Gradient descent $\mathscr{L}(\Theta)$

subject to $\|\theta_i\| = 0, \quad \forall i \in \mathscr{D},$

$\|\theta_i\| \geq 0, \quad \forall i \in \mathscr{A}.$

Update residual and if necessary return collapsed neurons to dormant set.

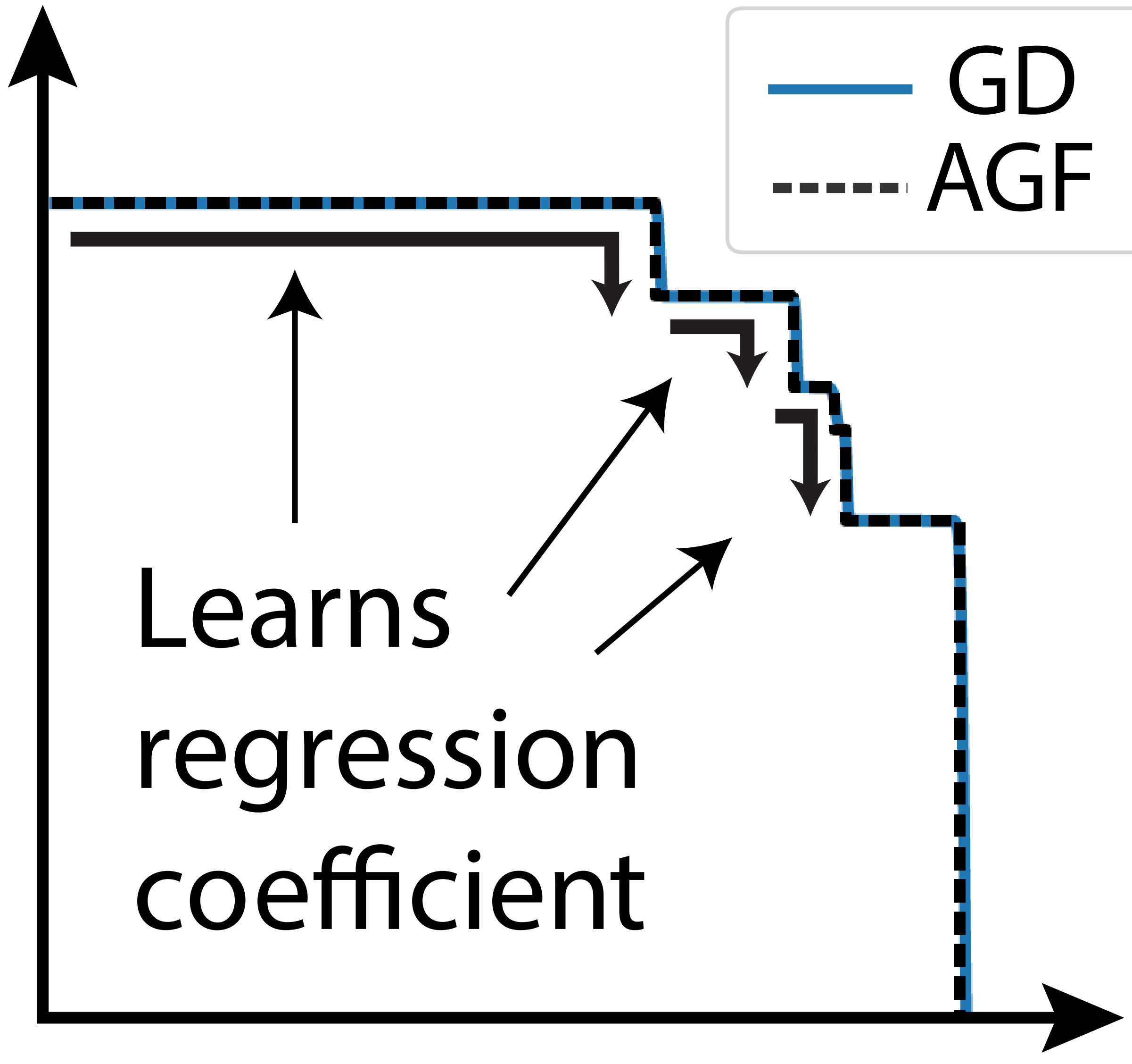
Termination: When $\mathscr{D} = \{\}$ or $r(x) = 0$



Part 2: AGF unifies many prior analysis of saddle-to-saddle dynamics

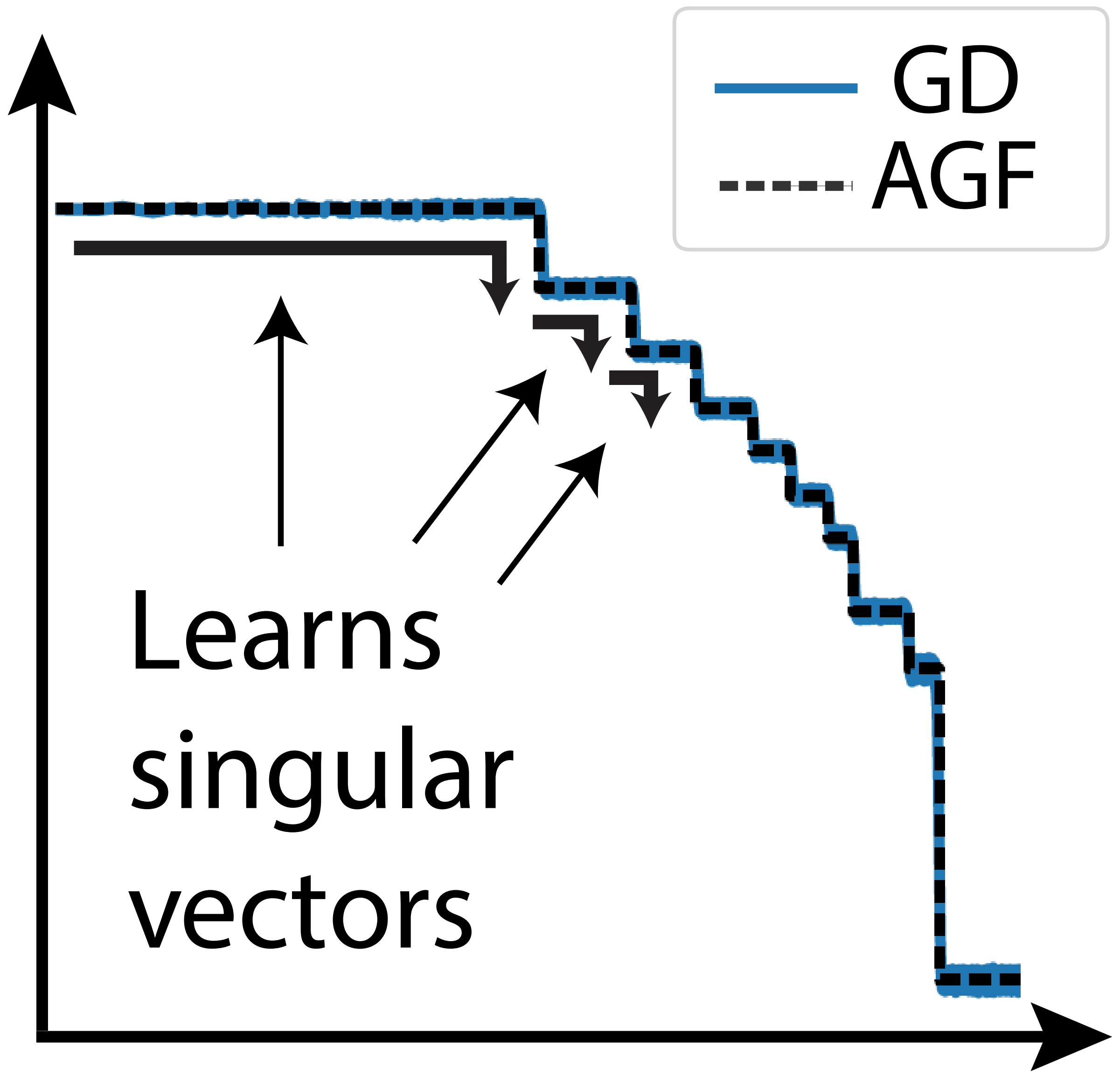
Diagonal Linear Networks

(Pesme & Flammarion, 2023)



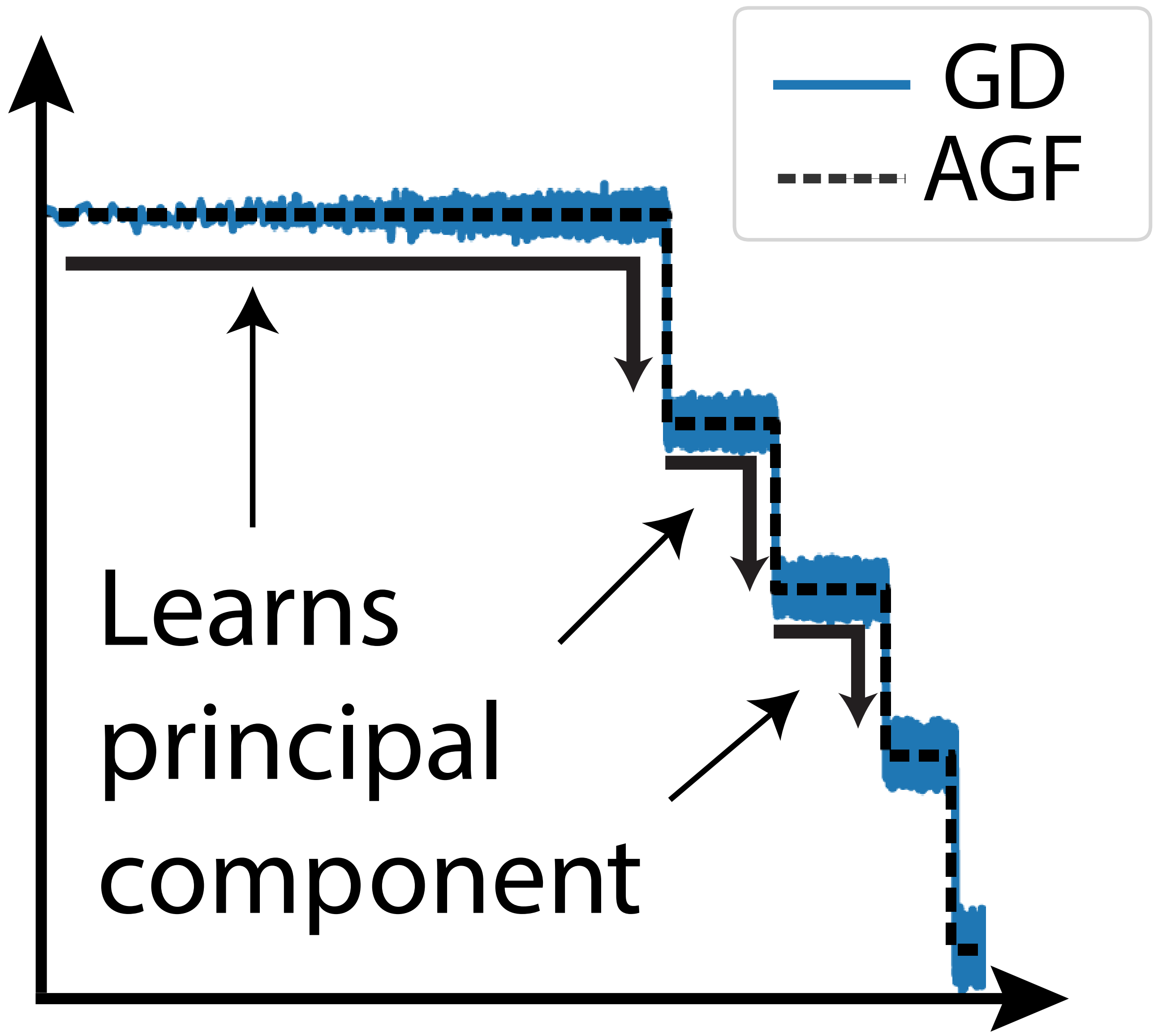
Fully Connected Linear Networks

(Saxe et al., 2014, Gidel et al., 2019, Li et al., 2020)



Attention-only Linear Transformers

(Zhang et al., 2025)



Part 3: AGF Explains the Emergence of Fourier Features in Modular Addition

PROGRESS MEASURES FOR GROKKING VIA MECHANISTIC INTERPRETABILITY

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How Do Machines 'Grok' Data?

By apparently overtraining them, researchers have seen neural networks discover novel solutions to problems.

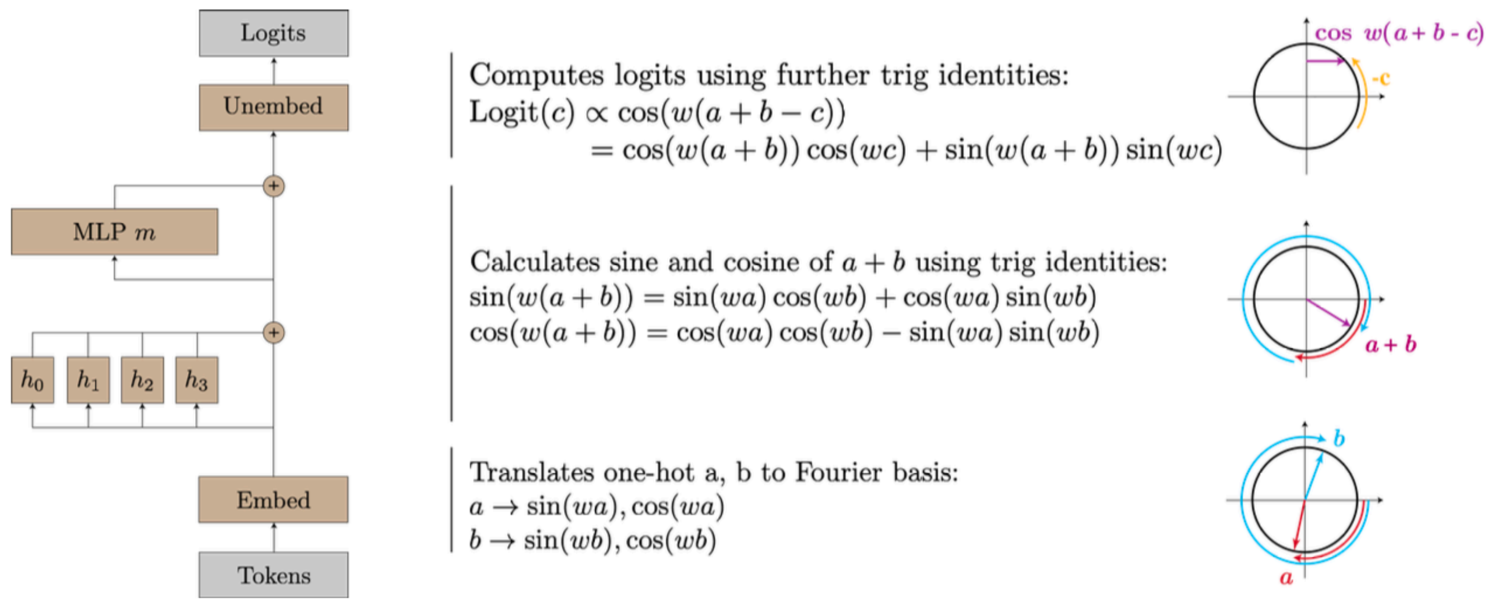
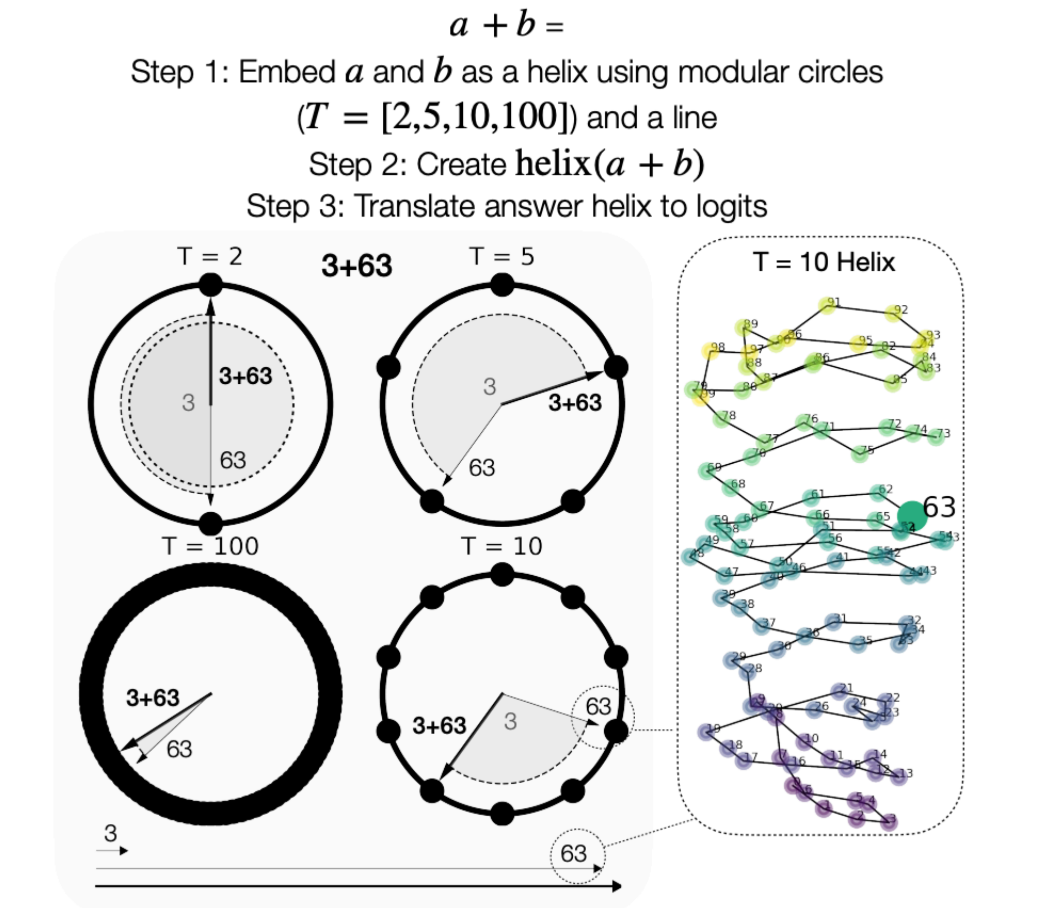


Figure 1: The algorithm implemented by the one-layer transformer for modular addition. Given two numbers a and b, the model projects each point onto a corresponding rotation using its embedding matrix. Using its attention and MLP layers, it then composes the rotations to get a representation of $a+b \mod P$. Finally, it "reads off" the logits for each $c \in \{0, 1, ..., P-1\}$, by rotating by $-c$ to get $\cos(w(a+b-c))$, which is maximized when $a+b \equiv c \mod P$ (since w is a multiple of $\frac{2\pi}{P}$).



Pre-trained Large Language Models Use Fourier Features to Compute Addition

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Language Models Use Trigonometry to Do Addition

Subhash Kantamneni[1] **Max Tegmark**[1]

A striking example of feature learning, prior works have shown networks trained to perform **modular addition** learn internal **Fourier features** and use trigonometric identities to implement **addition as rotations** on the circle

1. Utility Maximization - neurons specialize to a dominant frequency

We prove that the utility-maximizing unit vectors are cosine waves at the dominant frequency of the encoding vector.

Theorem F.3. *Let ξ be a frequency that maximizes $|\hat{x}[k]|$, $k = 1, \dots, p-1$, and denote by s_x the phase of $\hat{x}[\xi]$. Then the unit vectors $\theta_* = (u_*, v_*, w_*)$ that maximize the utility function $\mathcal{U}(\theta; y)$ take the form*

$$\begin{aligned} u_*[a] &= \sqrt{\frac{2}{3p}} \cos\left(2\pi \frac{\xi}{p} a + s_u\right) \\ v_*[b] &= \sqrt{\frac{2}{3p}} \cos\left(2\pi \frac{\xi}{p} b + s_v\right) \\ w_*[c] &= \sqrt{\frac{2}{3p}} \cos\left(2\pi \frac{\xi}{p} c + s_w\right), \end{aligned} \tag{58}$$

where $a, b, c \in \{0, \dots, p-1\}$ are indices and $s_u, s_v, s_w \in \mathbb{R}$ are phase shifts satisfying $s_u + s_v \equiv s_w + s_x \pmod{2\pi}$. They achieve a maximal value of $\bar{\mathcal{U}}^* = \sqrt{2/(27p^3)}|\hat{x}[\xi]|^3$. Moreover, the utility function has no local maxima other than the ones described above.

2. Cost Minimization - group of neurons collaborate to minimize loss

We prove that frequency aligned neurons remain aligned and that a group of aligned neurons must distribute their phase shifts to remove that frequency from the residual

Theorem F.6. *We have the following lower bound:*

$$\mathcal{C}(\Theta) - \mathcal{U}_0(\Theta) \geq -\frac{|\hat{x}[\xi]|^2}{p}. \tag{78}$$

Moreover, equality holds if, and only if, we have that $\sum_{i=1}^N C_i \cos(\alpha_i) = \sum_{i=1}^N C_i \sin(\alpha_i) = 0$ for any choice of (C_i, α_i) among

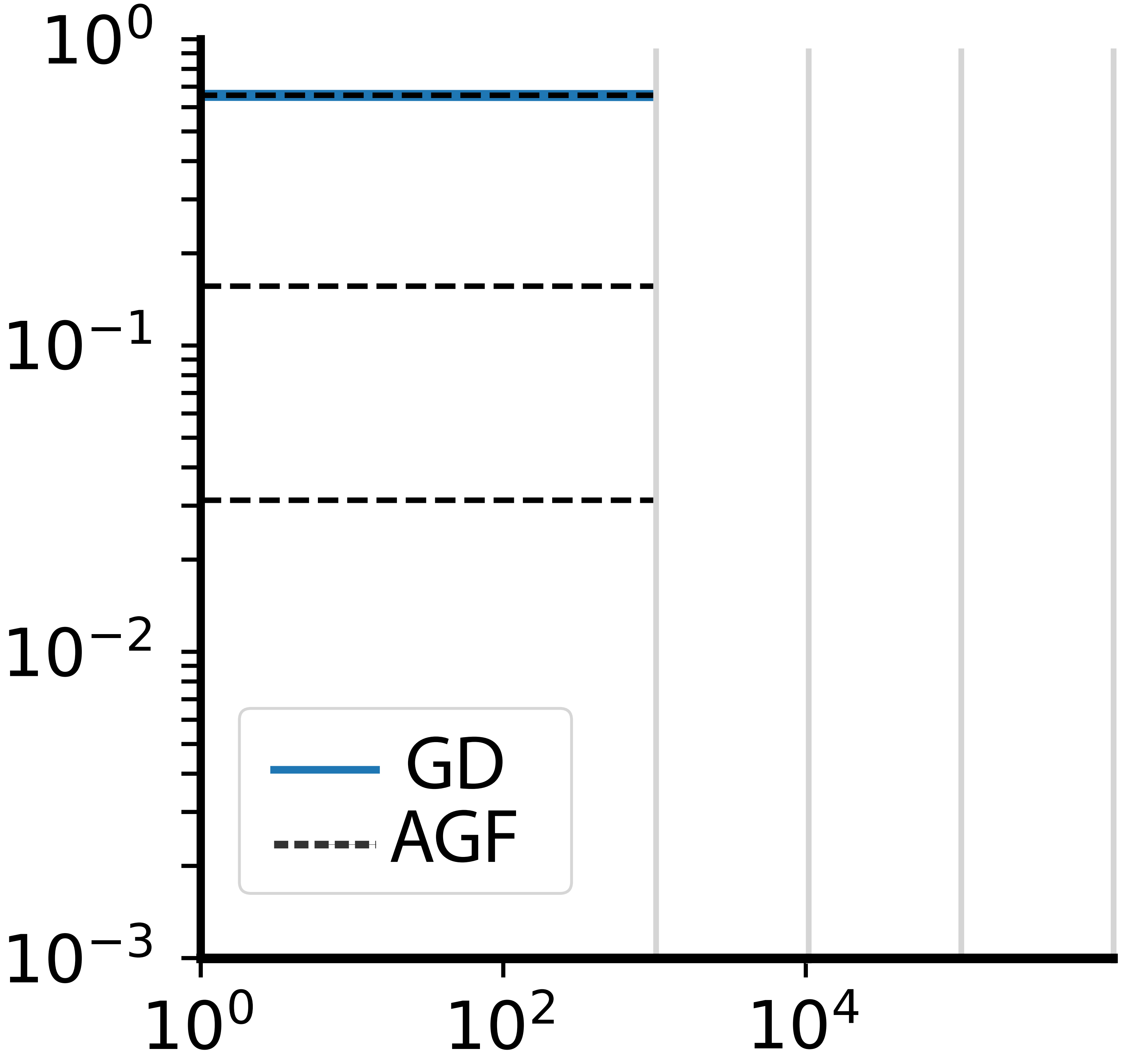
$$\begin{aligned} &(A_w^i((A_u^i)^2 + (A_v^i)^2), s_w^i), \\ &(A_w^i(A_u^i)^2, s_w^i \pm 2s_u^i), \\ &(A_w^i(A_v^i)^2, s_w^i \pm 2s_v^i), \\ &(A_w^i A_u^i A_v^i, s_w^i \pm (s_u^i - s_v^i)), \\ &(A_w^i A_u^i A_v^i, s_w^i + s_u^i + s_v^i), \end{aligned} \tag{79}$$

and, moreover, $\sum_{i=1}^N A_w^i A_u^i A_v^i \sin(s_w^i + s_x - s_u^i - s_v^i) = 0$ and $\sum_{i=1}^N A_w^i A_u^i A_v^i \cos(s_w^i + s_x - s_u^i - s_v^i) = \sqrt{54p}/|\hat{x}[\xi]|$.

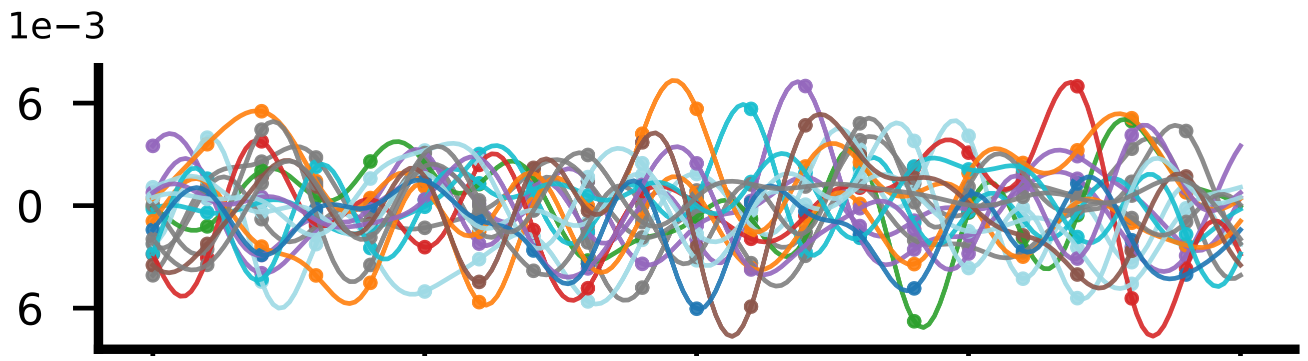
Putting it together: Greedy Fourier decomposition

Parameter Space Explanation

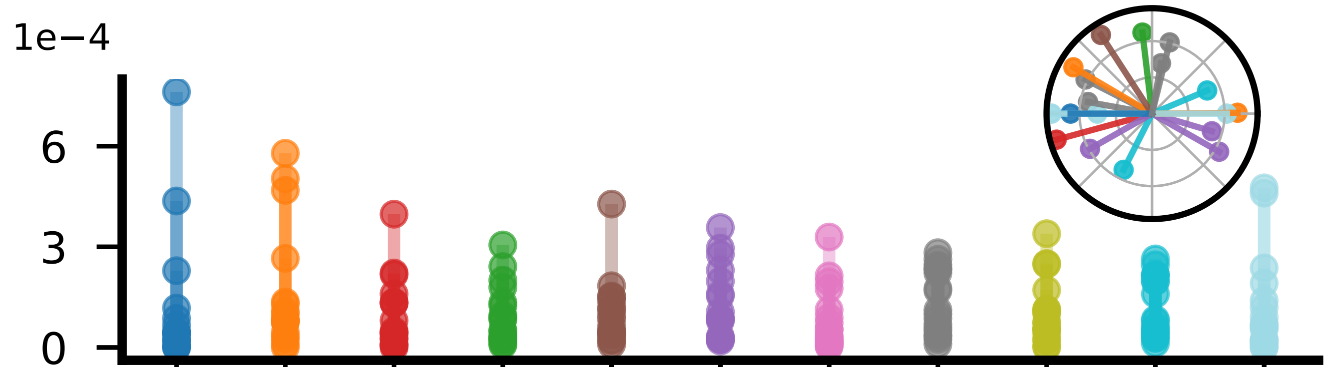
Train Loss



Weights w_i

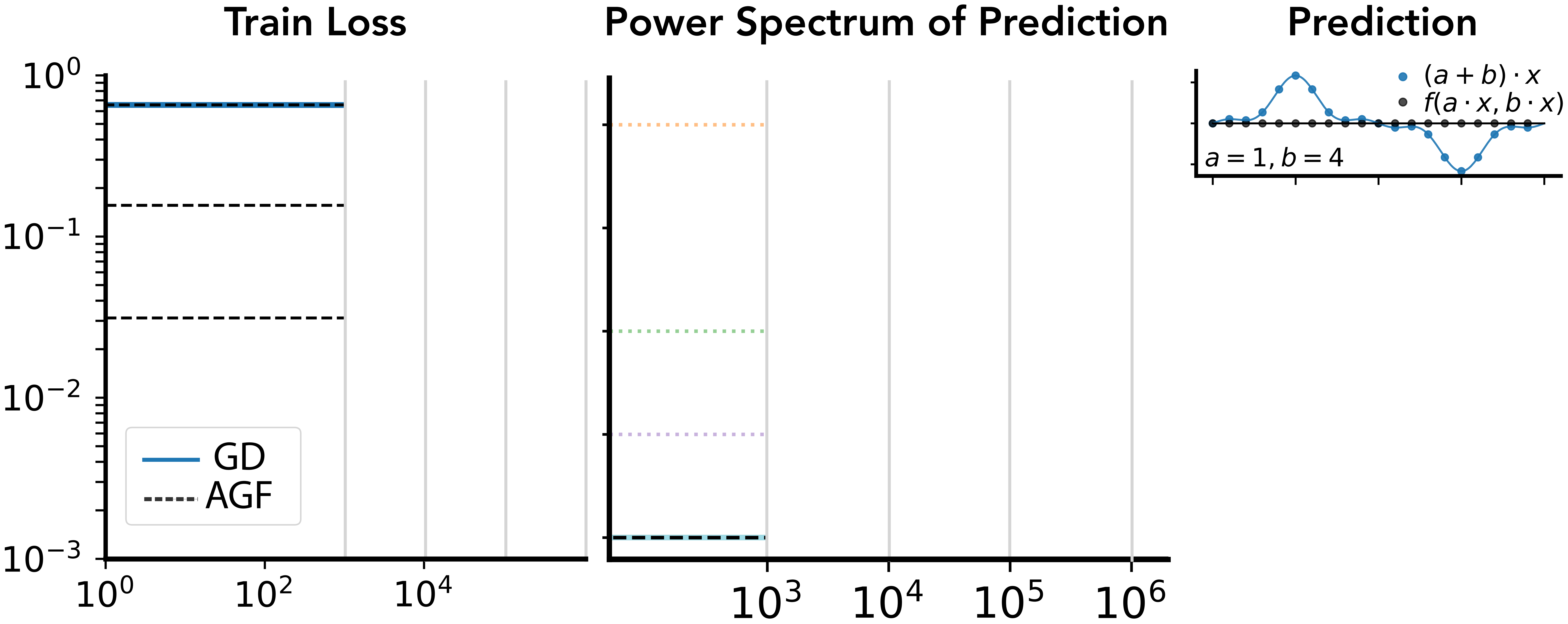


Frequency and Phase



Putting it together: Greedy Fourier decomposition

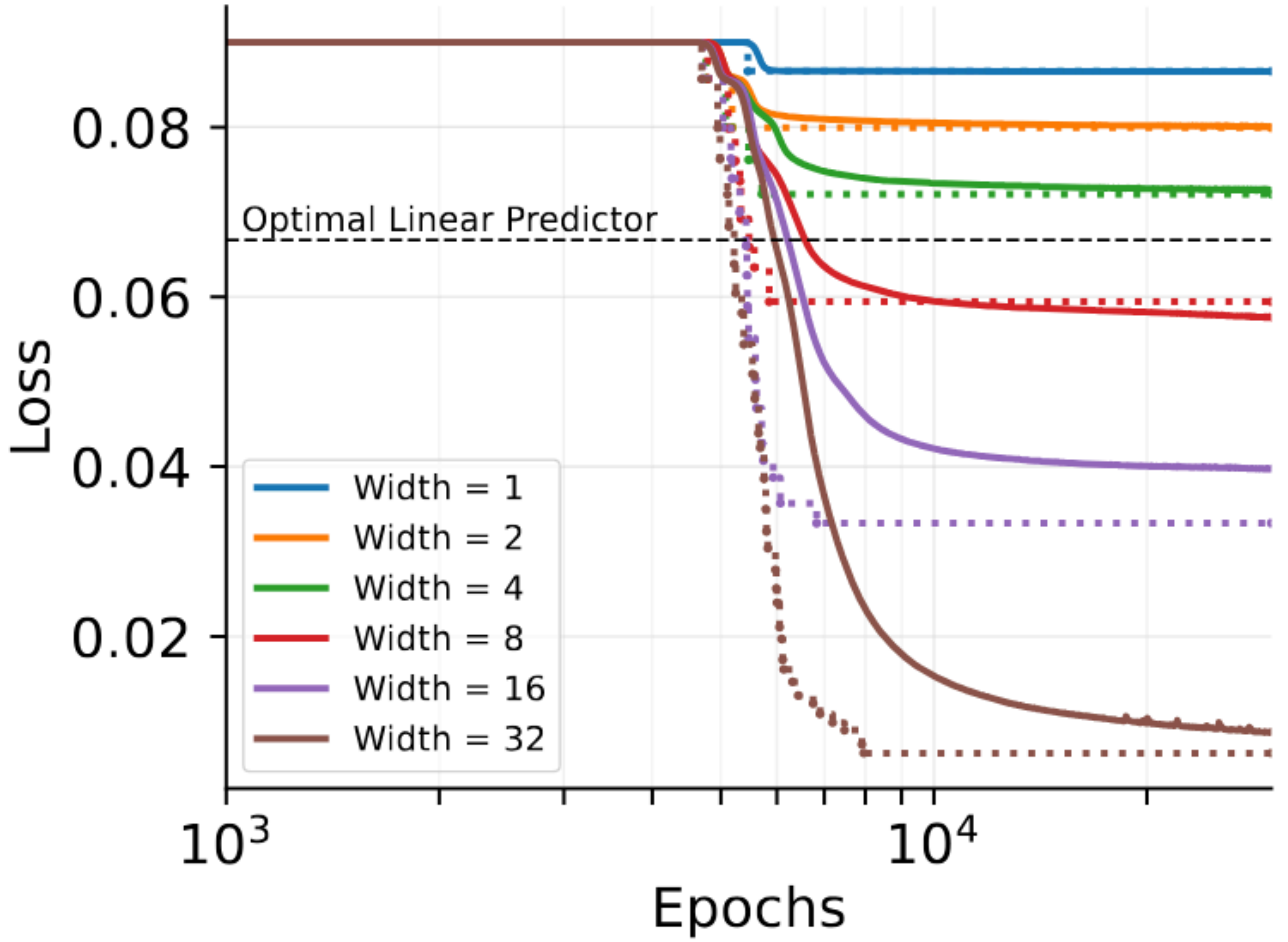
Function Space Perspective



Limitations of AGF and directions for future work

- On **natural data tasks**, loss curves are often not visibly stepwise even at very small initialization scales
  - Cost min steps bleed together
  - Many small steps lead to a smooth decay (emergent behavior and scaling laws)

Two-layer ReLU network on a subset of CIFAR-10



- AGF is an **ansatz for gradient flow** — can we prove a general conjecture (over a class of problems)

- Can we connect AGF to more classic feature learning analysis in **multi-index** models and **teacher–student** settings

- Can we extend the ideas of AGF to **deeper networks**? What is a “neuron”?

Thank you!

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Poster Session: Thurs. Dec. 4th 11am-2pm

Exhibit Hall C,D,E, San Diego, CA