Nonlinear Laplacians

Tunable principal component analysis under directional prior information

Yuxin Ma, Dmitriy (Tim) Kunisky
Department of Applied Mathematics and Statistics, Johns Hopkins University
{yma93, kunisky}@jhu.edu

Graph Planted Clique Problem

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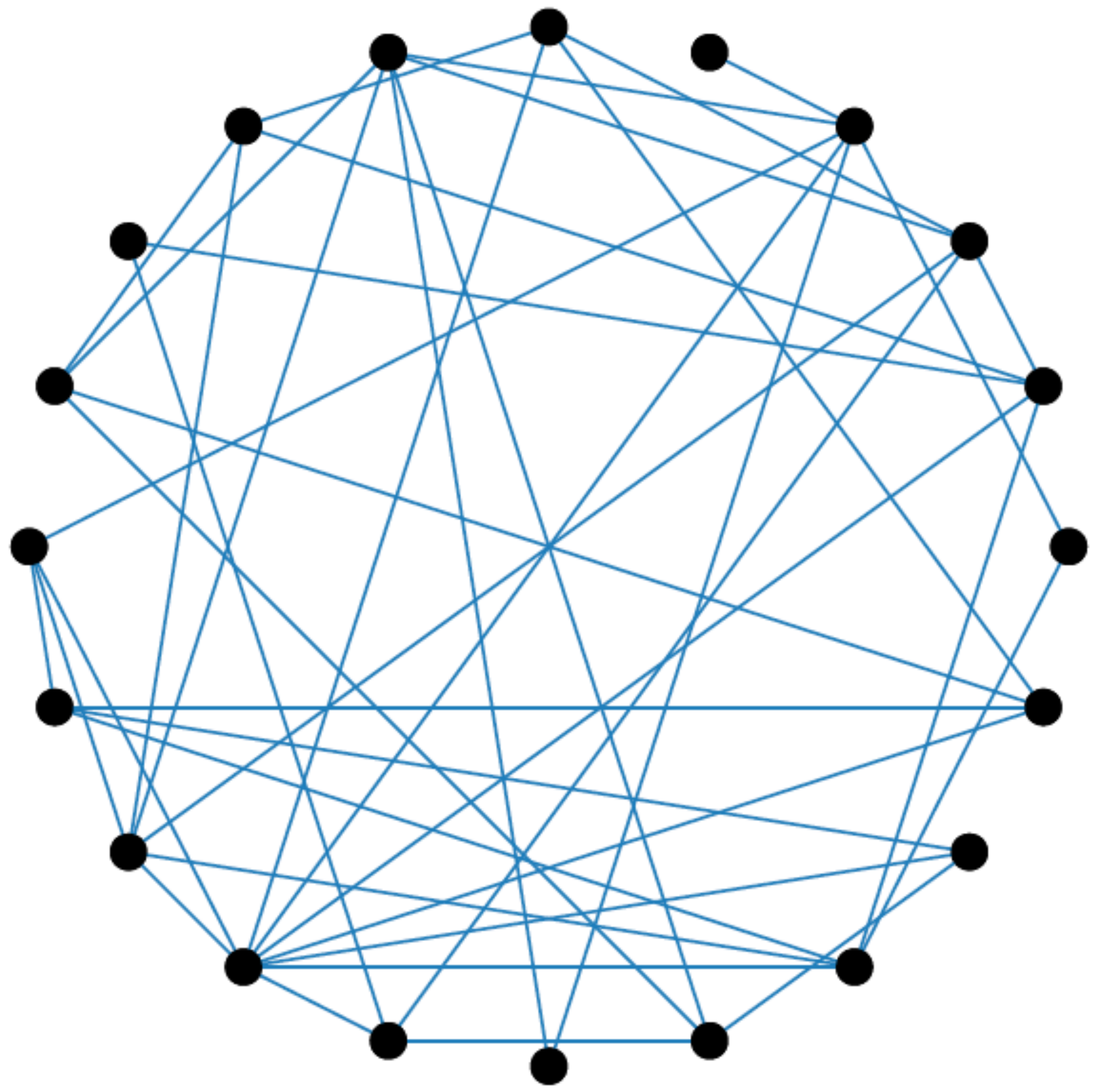

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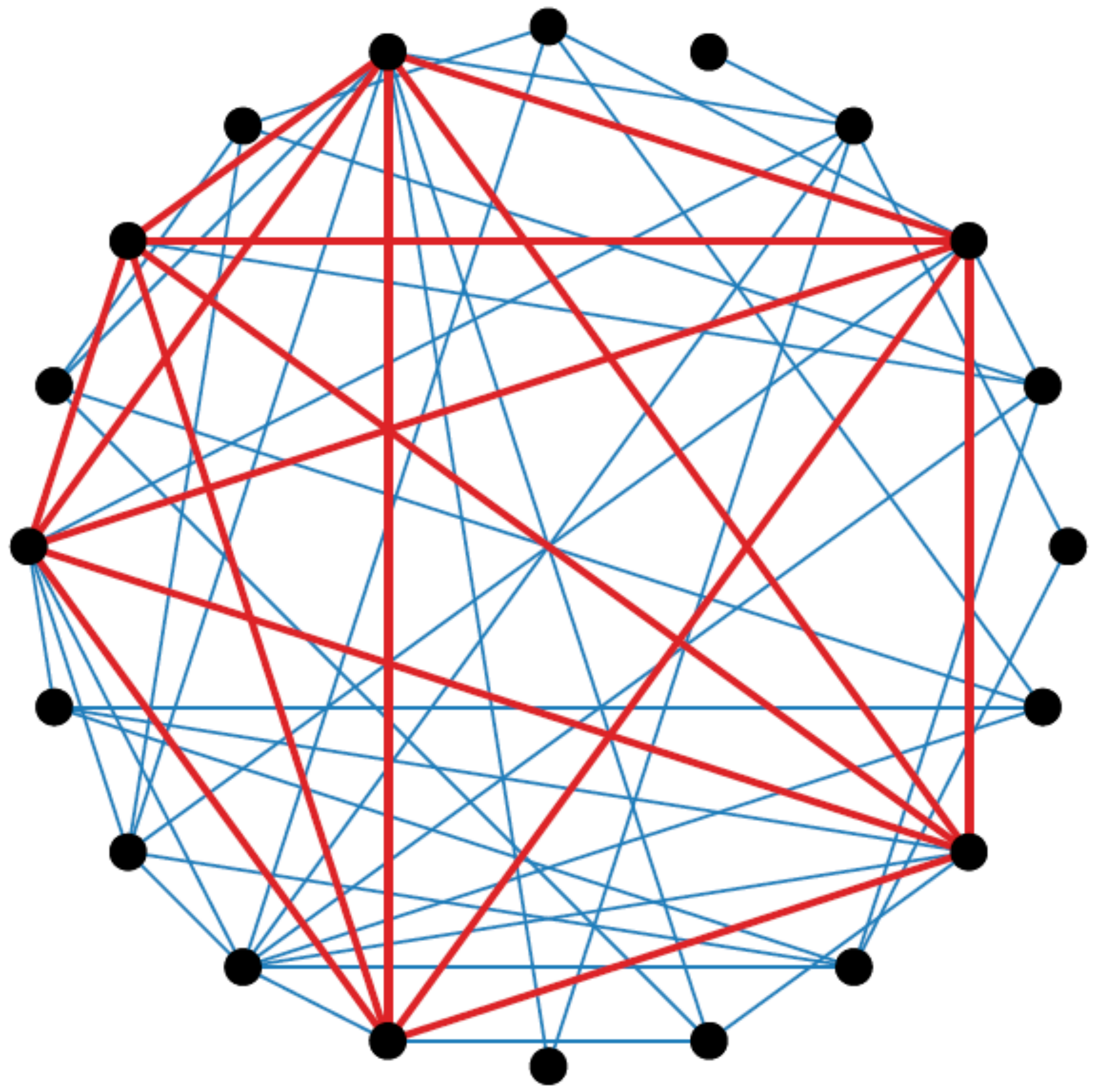

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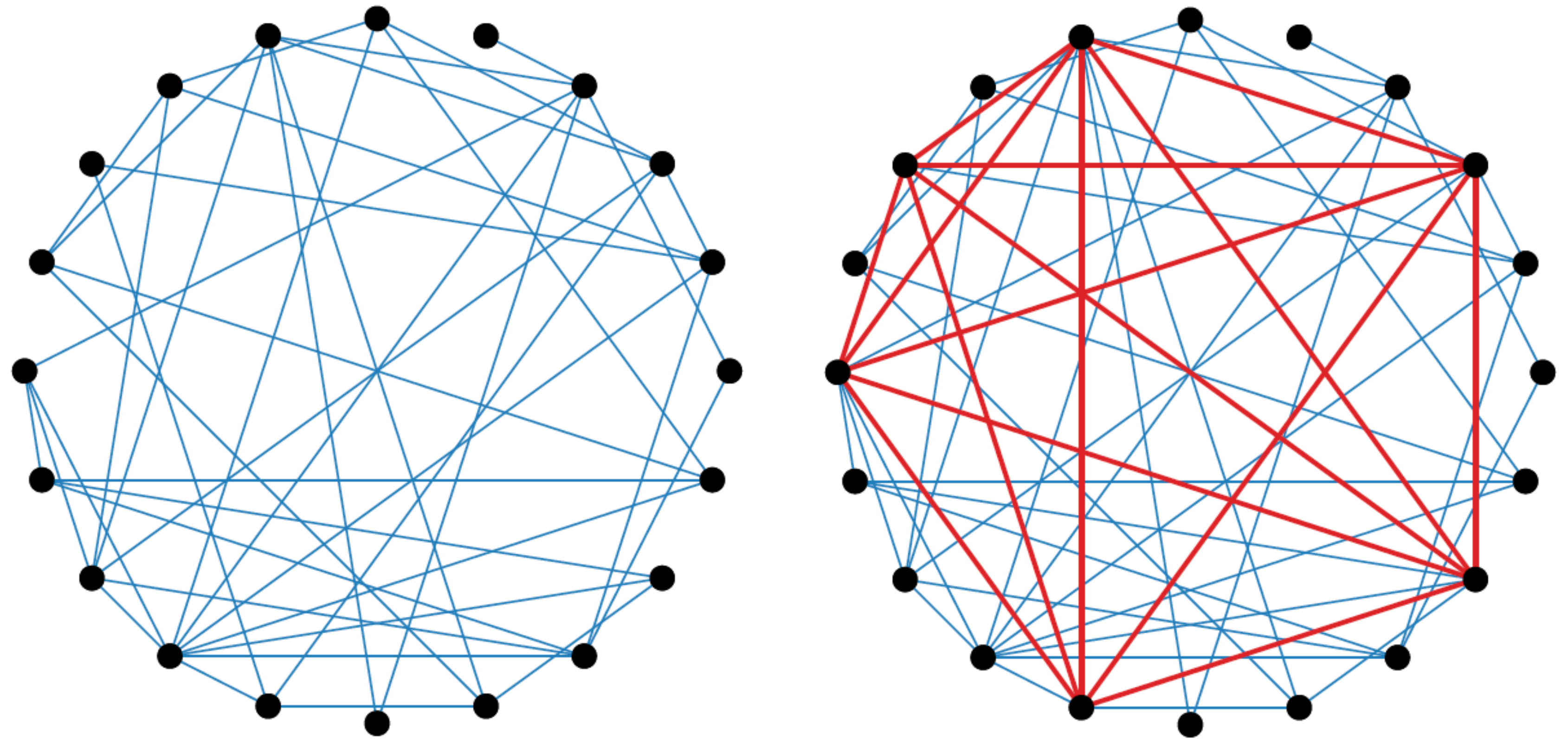
Want to either

- **Detection:** decide whether $S = \varnothing$. (Whether a clique is planted).
- **Estimation:** produce $\hat{S} \approx S$. (Recover the clique).



Naive spectral algorithm

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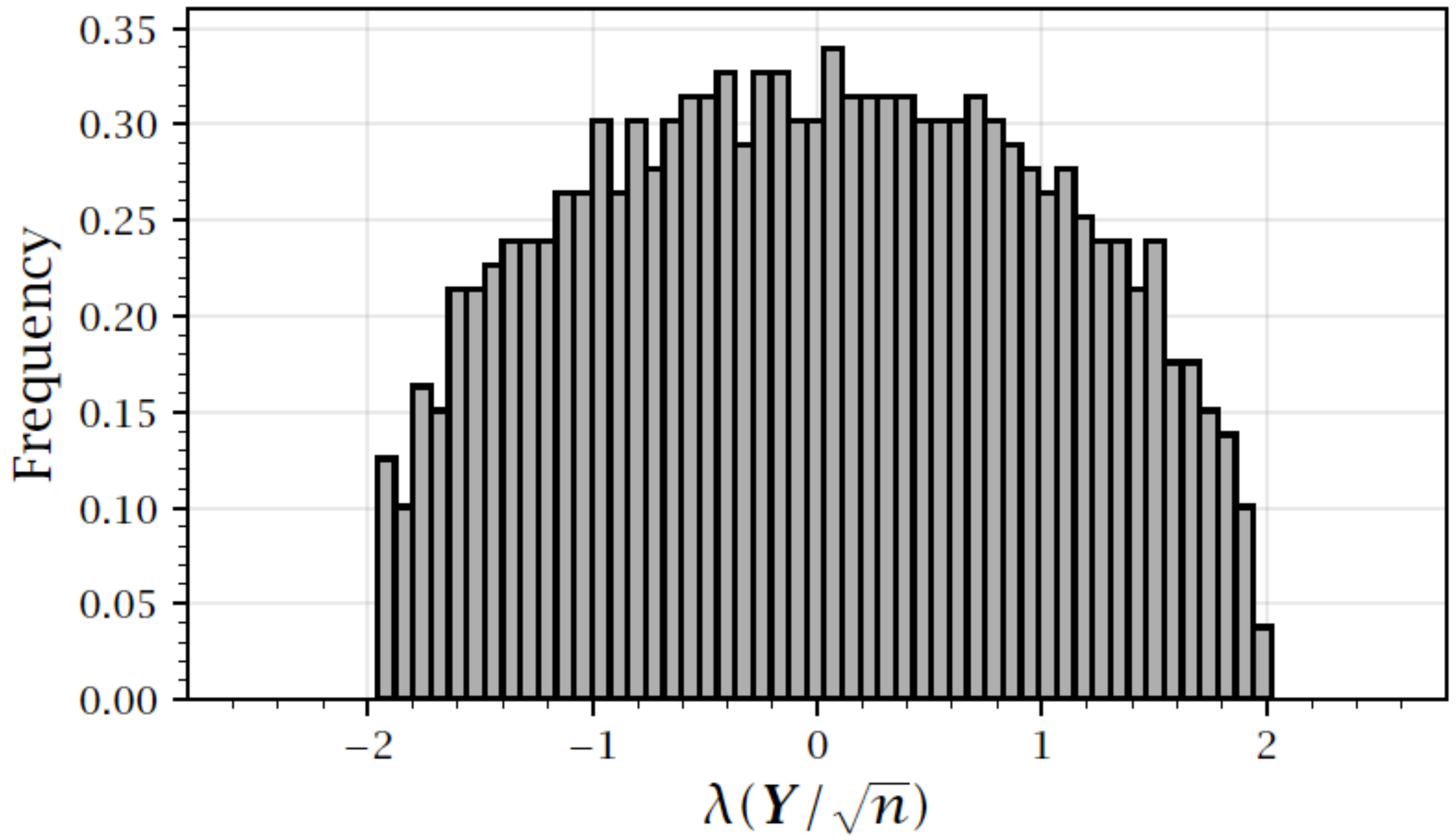

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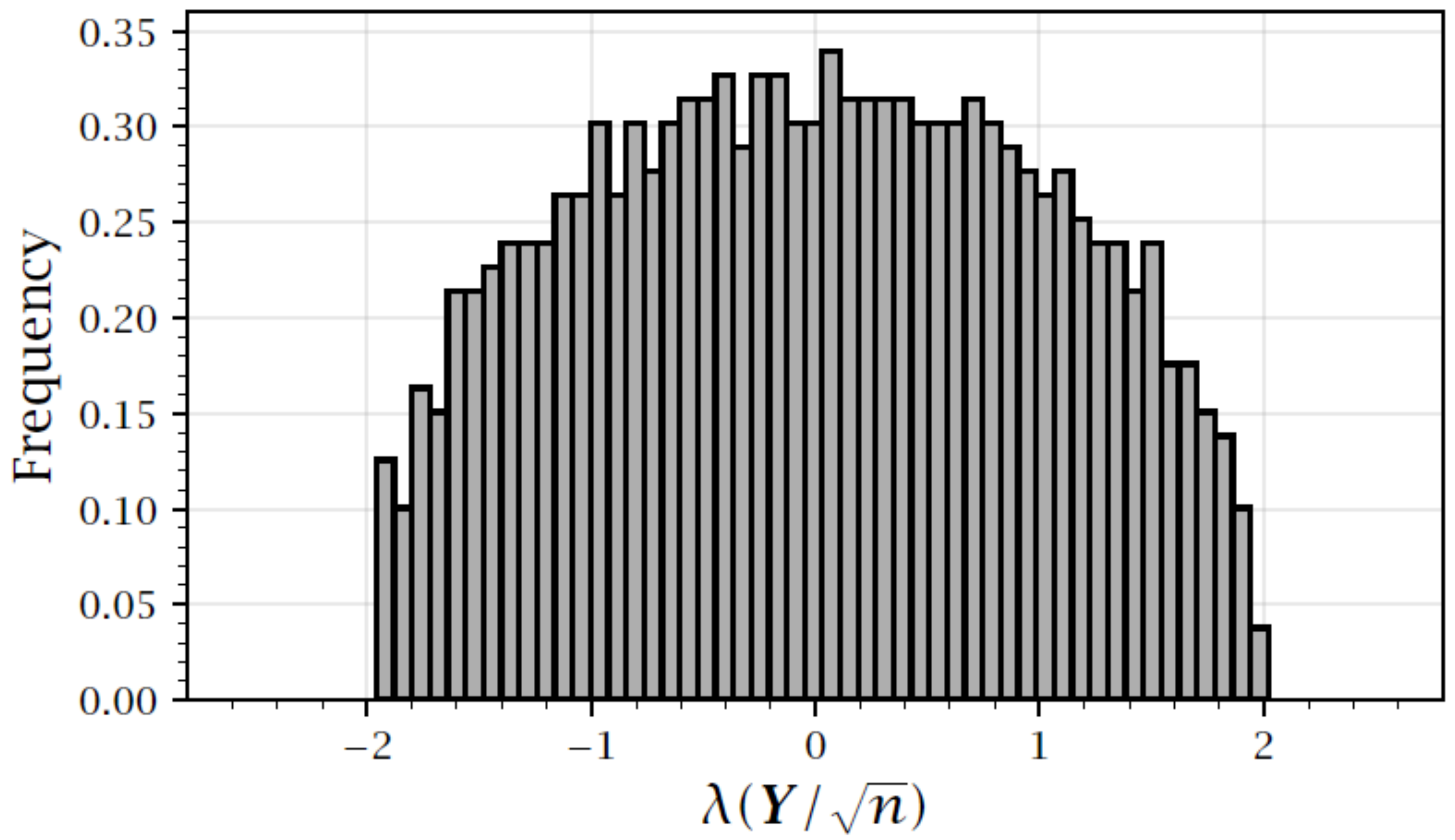

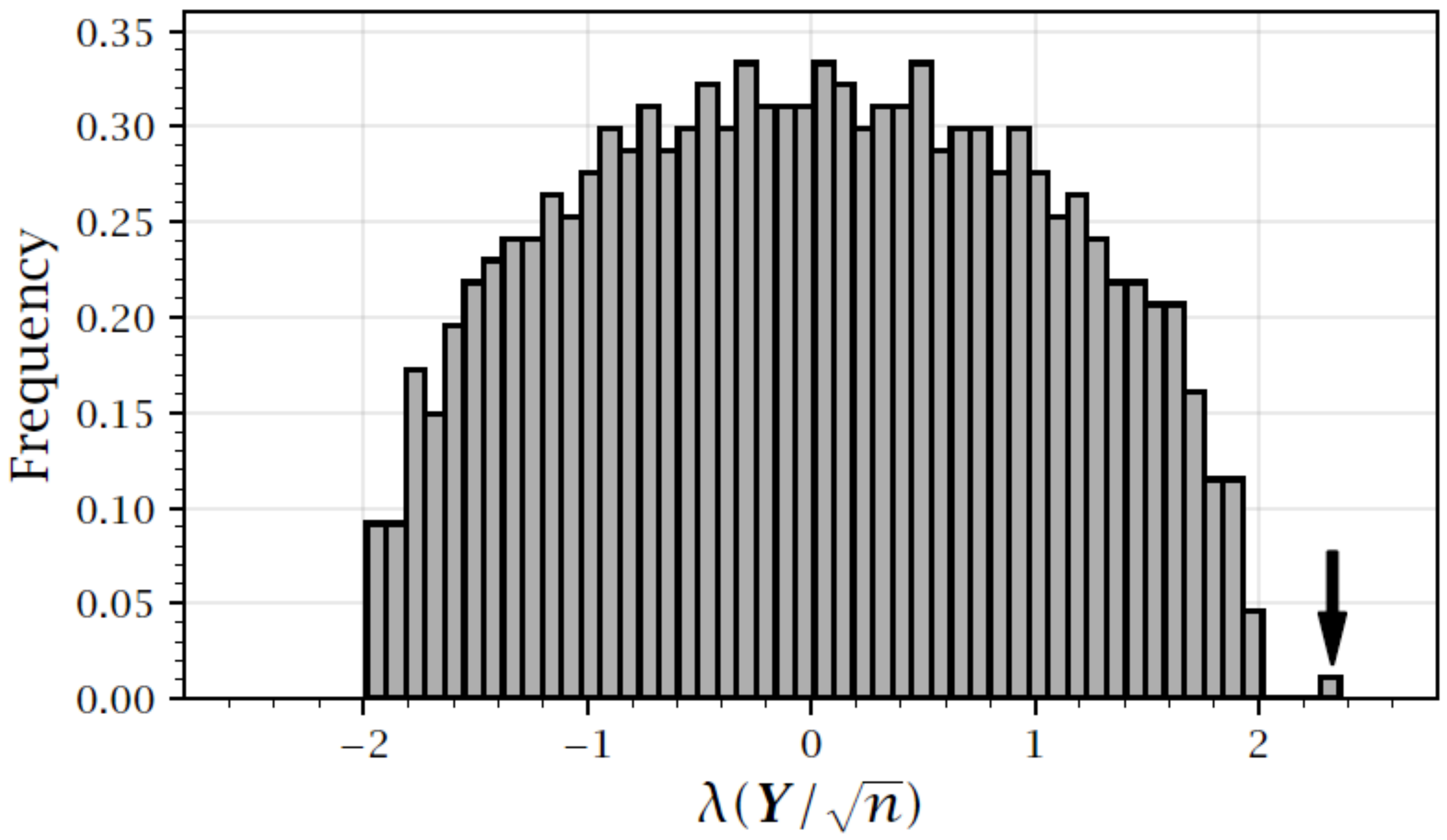

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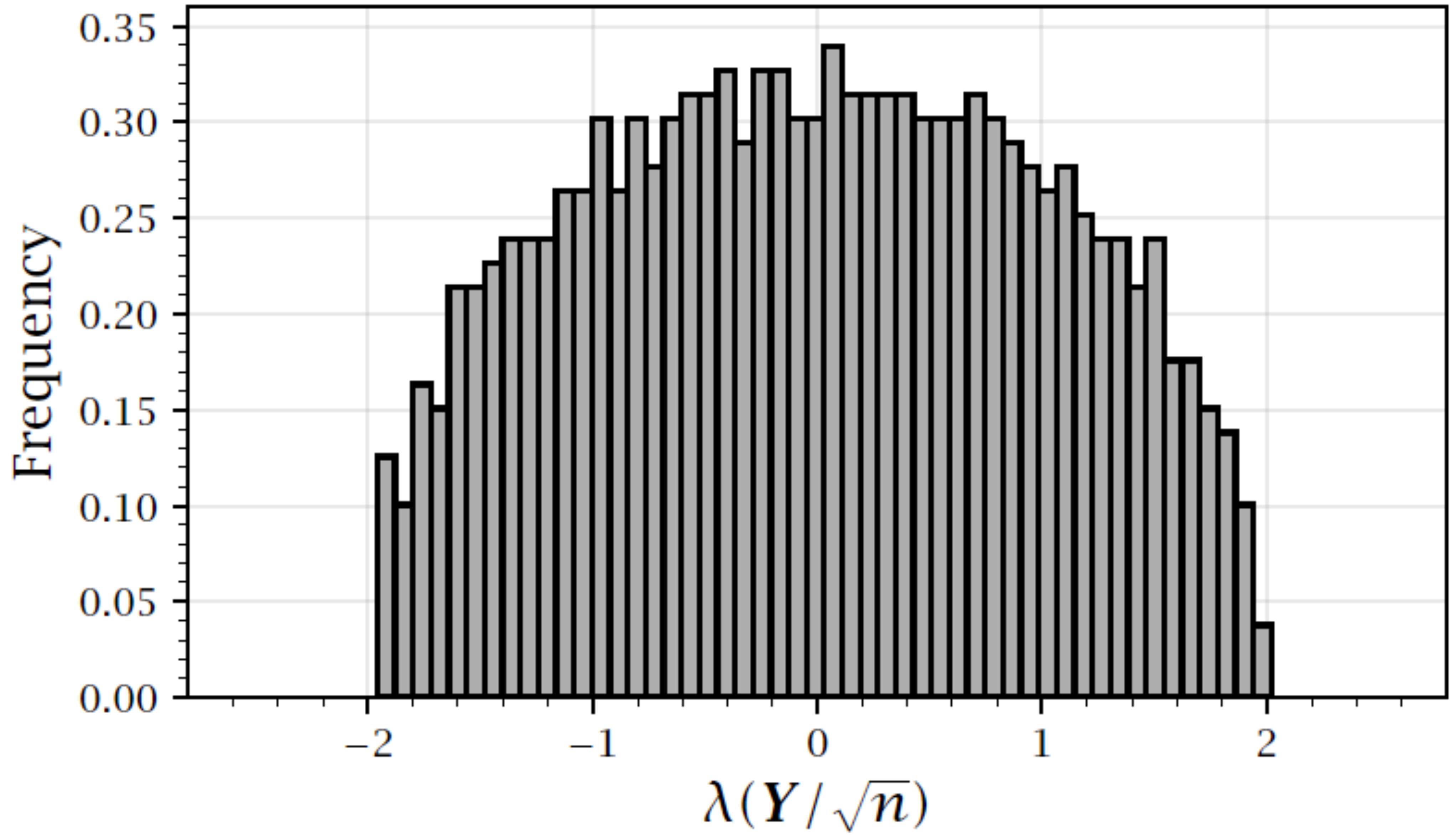
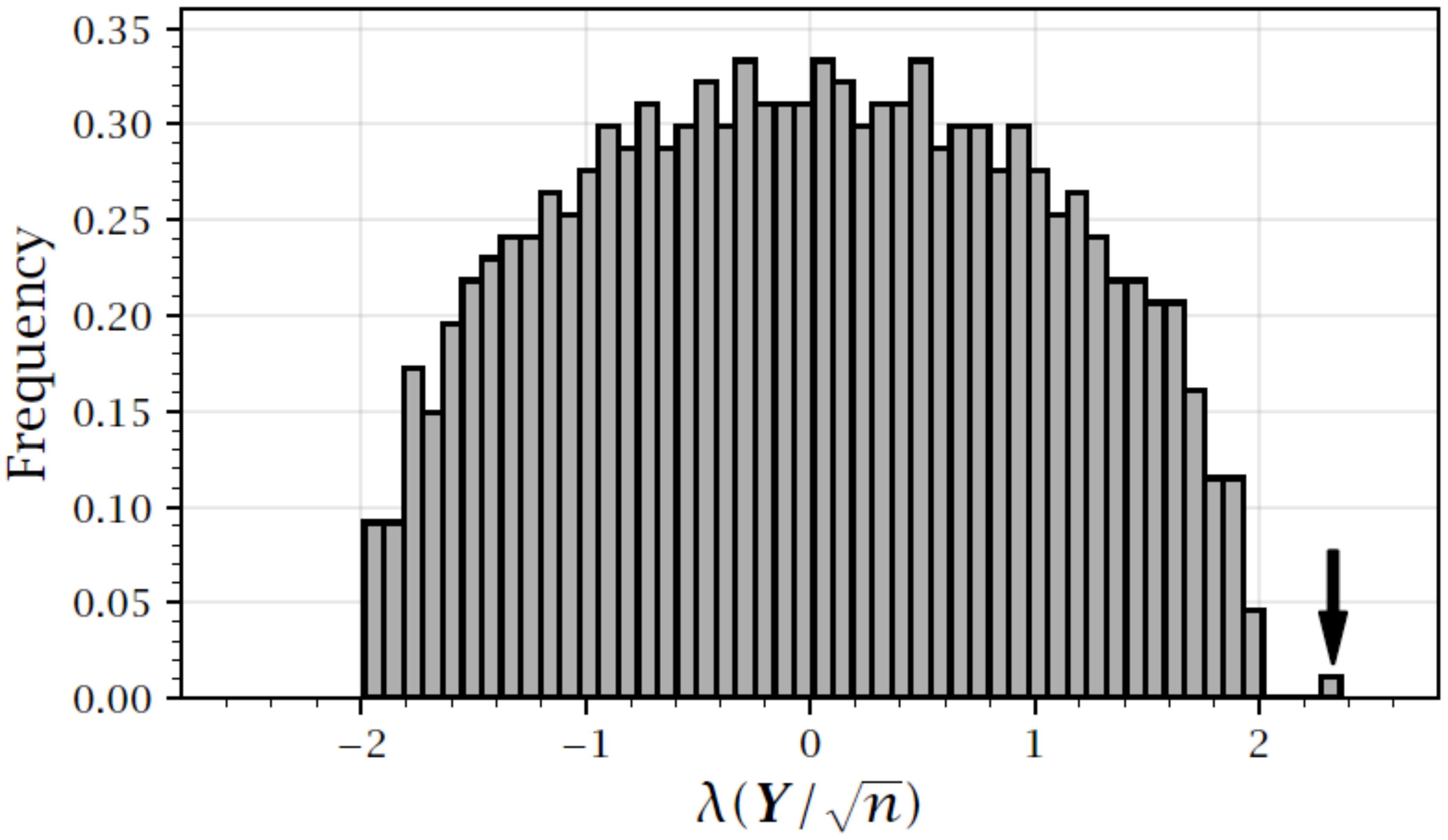
- Outlier eigenvalue $\iff$ $|S| > \sqrt{n}$





Naive spectral algorithm

- **Detection**: decide there's a planted clique if $\lambda_1\left(\frac{\mathbf{Y}}{\sqrt{n}}\right) > 2 + \varepsilon$
- **Estimation**: $v_1\left(\frac{\mathbf{Y}}{\sqrt{n}}\right) \approx \frac{\mathbf{1}_S}{\|\mathbf{1}_S\|}$

Non-trivial detection and estimation $\iff |S| > \sqrt{n}$

Our proposal: Nonlinear Laplacians

The naive spectral algorithm discards crucial prior information:
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Idea: Use λ_1, v_1 of the Nonlinear Laplacian matrix

$$\mathbf{L}_\sigma(\mathbf{Y}) = \hat{\mathbf{Y}} + \mathrm{diag}(\sigma(\hat{\mathbf{Y}}\mathbf{1})), \quad \hat{\mathbf{Y}} = \frac{\mathbf{Y}}{\sqrt{n}}$$

- $\hat{\mathbf{Y}}\mathbf{1}$ captures the degree information.
- $\sigma : \mathbb{R} \to \mathbb{R}$ bounded and monotone, applied entry-wise.

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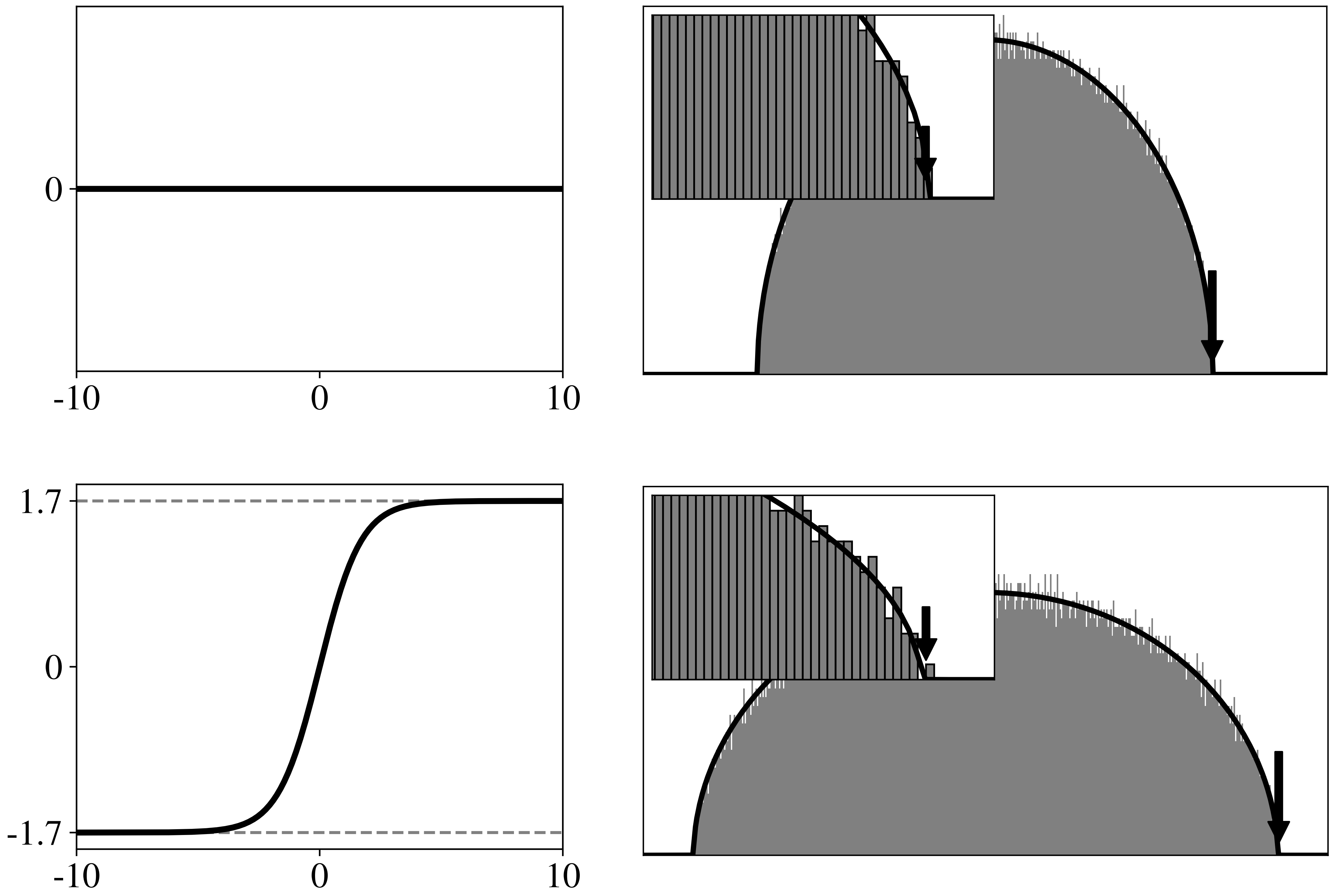
Nonlinearity σ Eigenvalues: $|S| = 0$



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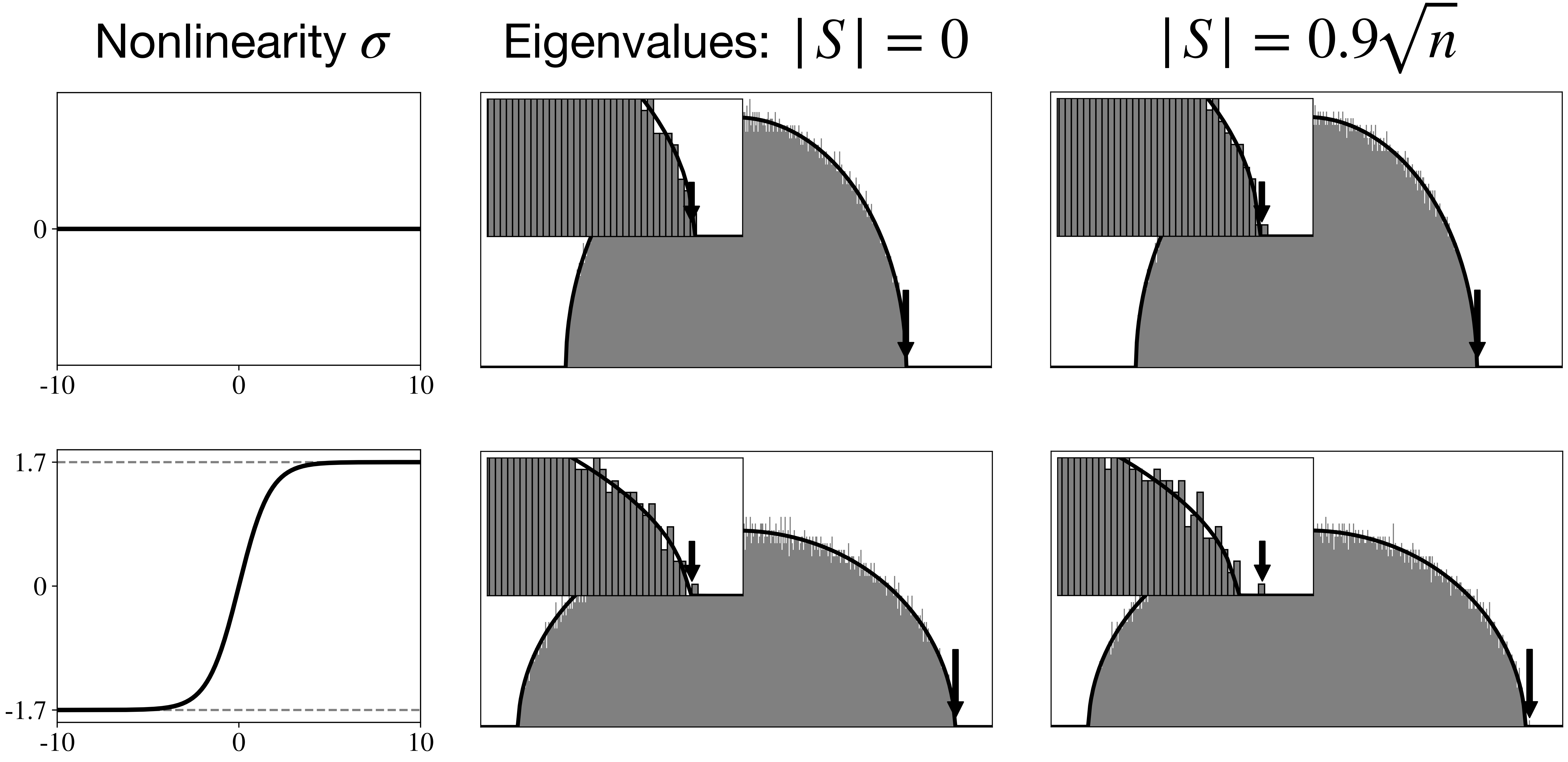

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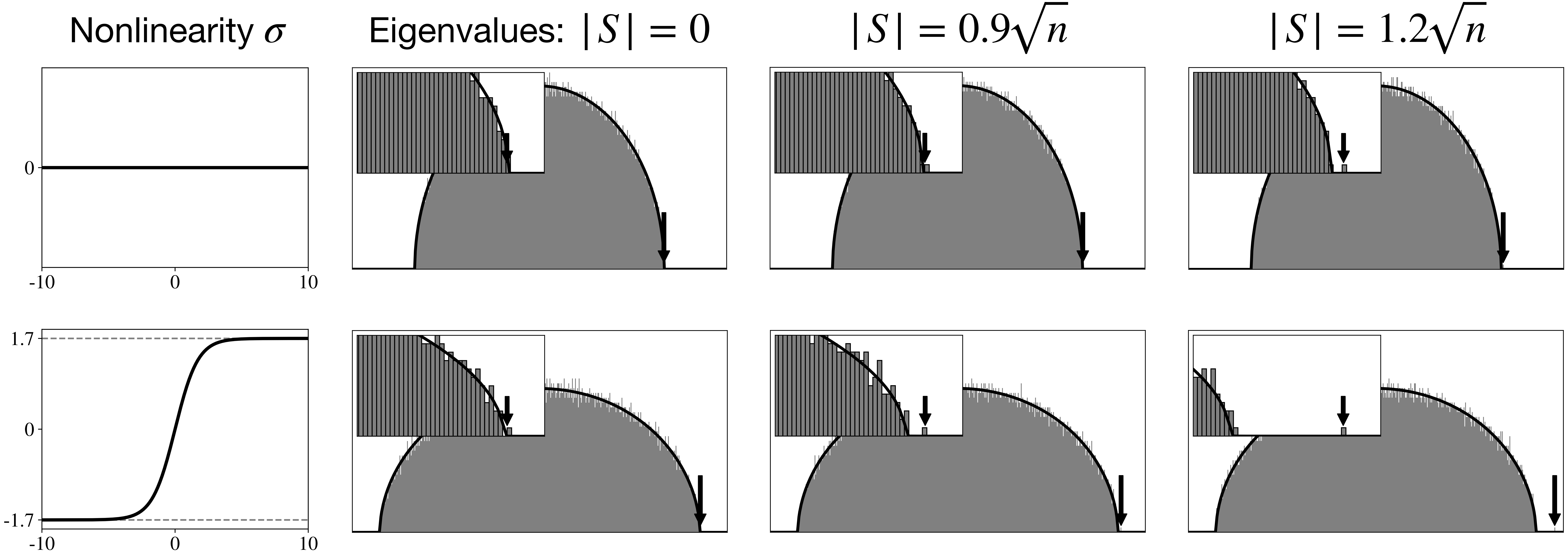

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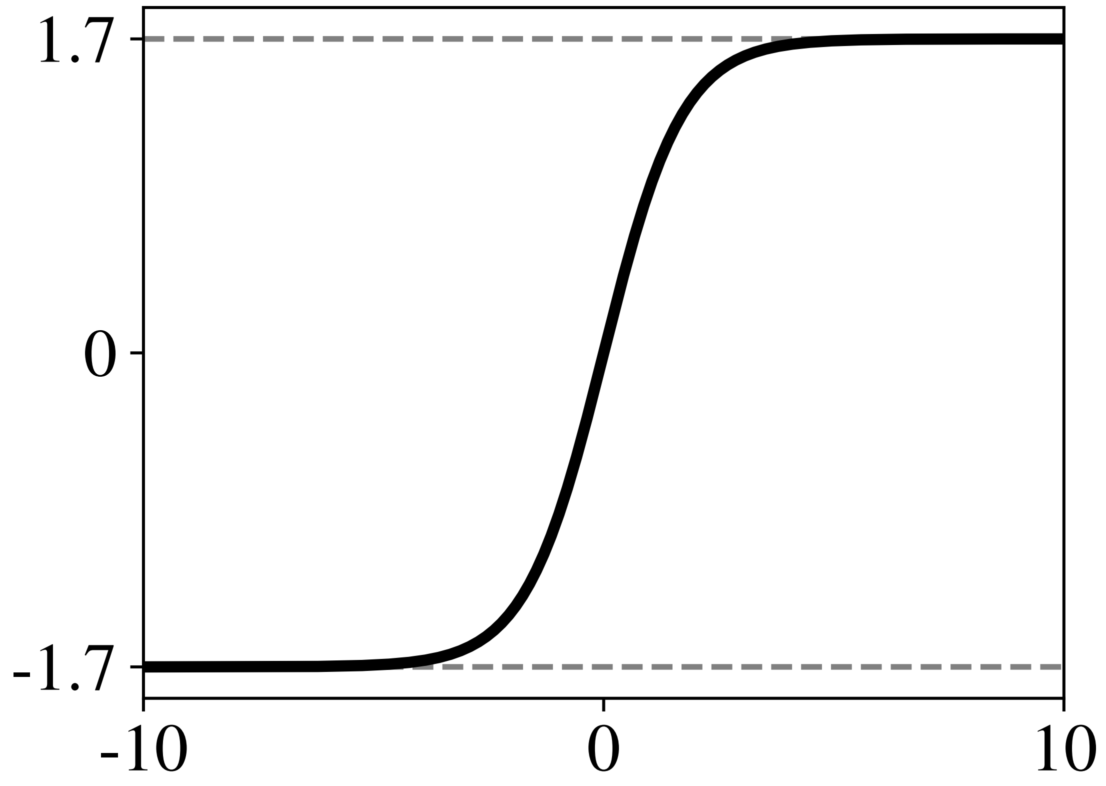
$\sigma =$



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- For any clique size $< \sqrt{n}$, neither the spectral information alone nor the degree information alone cannot detect.
- By combining them in this simple way, we achieve detection for any clique size $> 0.76\sqrt{n}$.

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Poster on Thursday 4 Dec, 4:30 - 7:30 pm PST.