Graph Alignment via Birkhoff Relaxation

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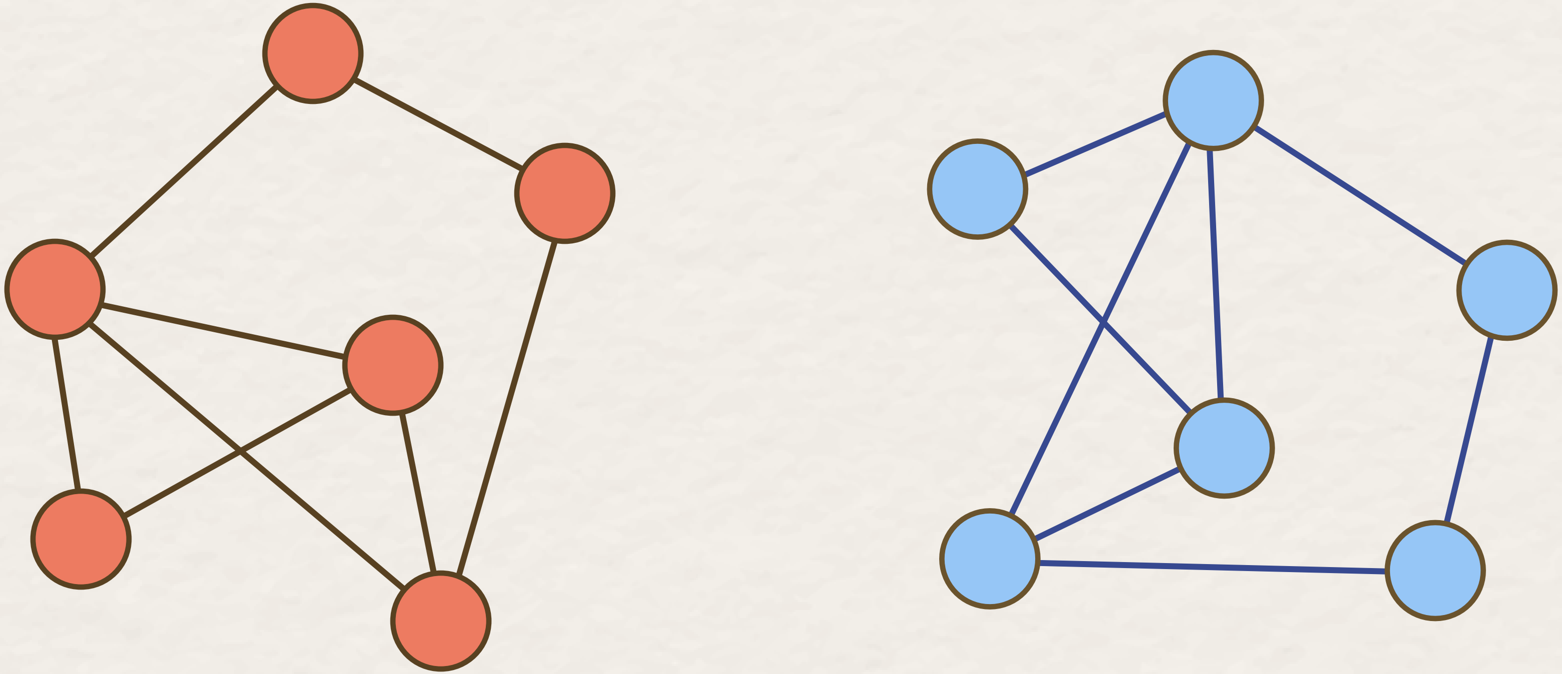
Graph Alignment

- Find a **vertex correspondence** so that the **edge overlap** is maximized



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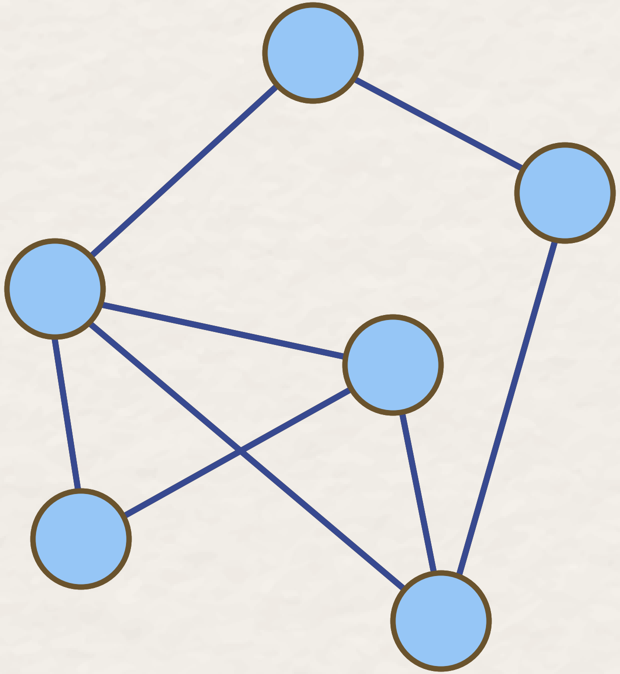
- Special Case: Graph Isomorphism (both graphs are exactly the same)



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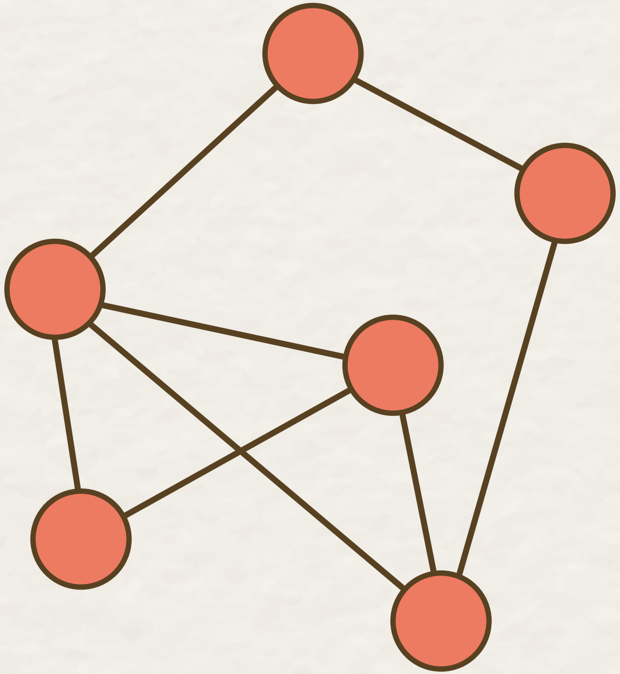
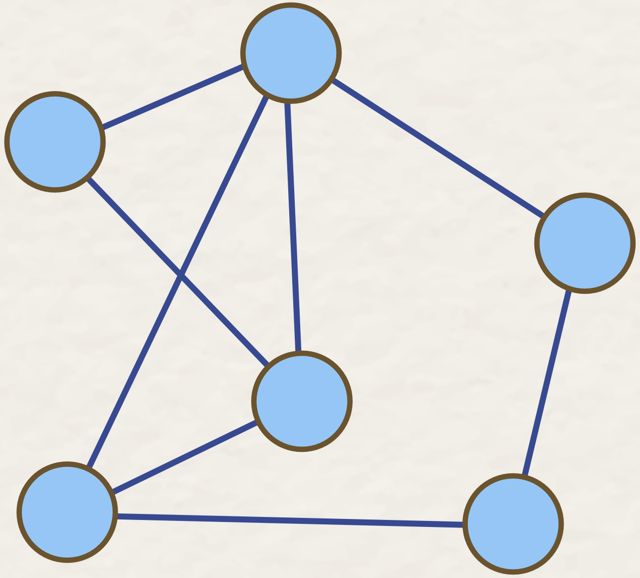
- The graphs need not be equal -> we can still ask the same question!





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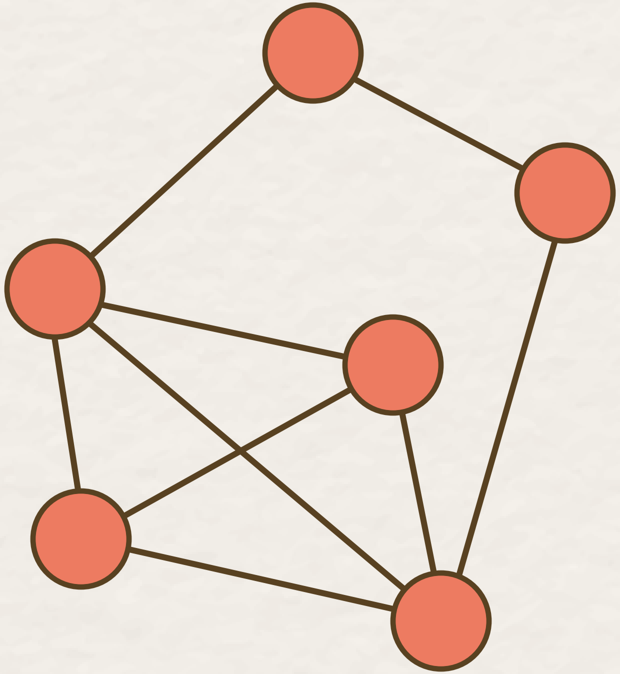

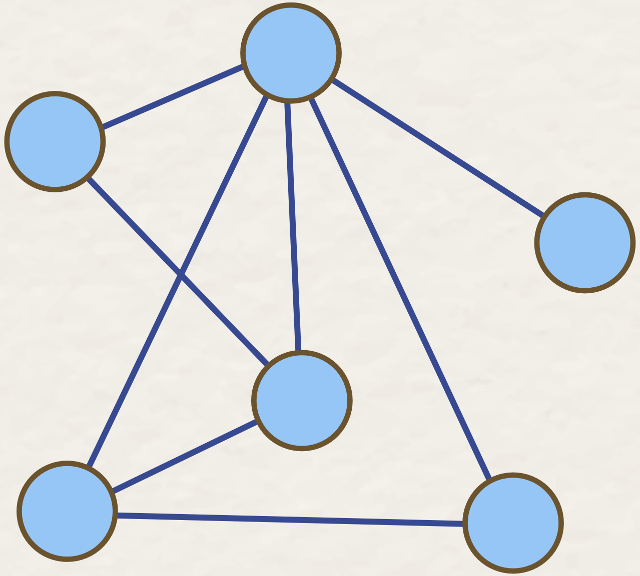

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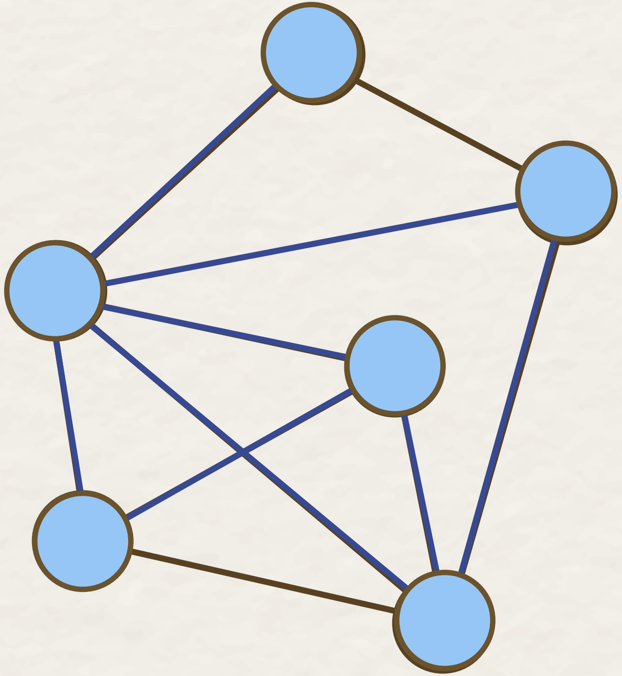
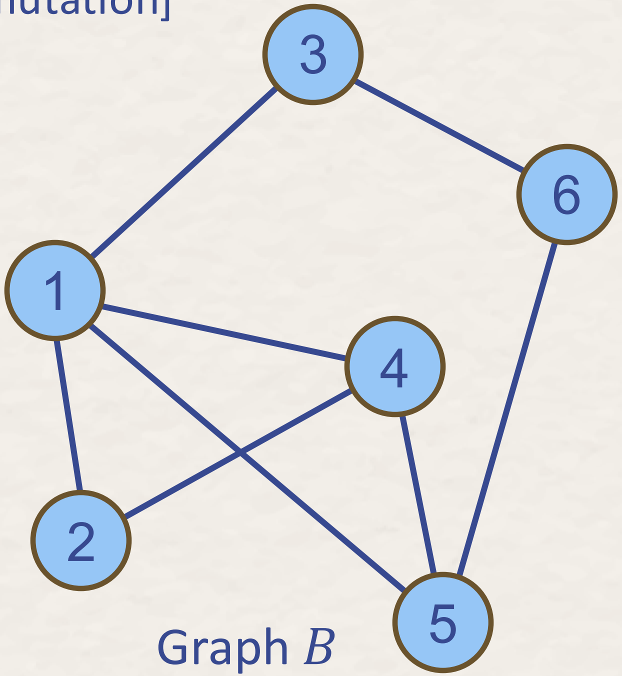


Graph Alignment as QAP

- Given adjacency matrices $A, B \in \mathbb{R}^{n \times n}$
- We want to relabel the vertices of $B : i \to \pi(i)$ [Permutation]



Graph B

Graph Alignment as QAP

- Given adjacency matrices $A, B \in \mathbb{R}^{n \times n}$
- We want to relabel the vertices of $B : i \to \pi(i)$ [Permutation]

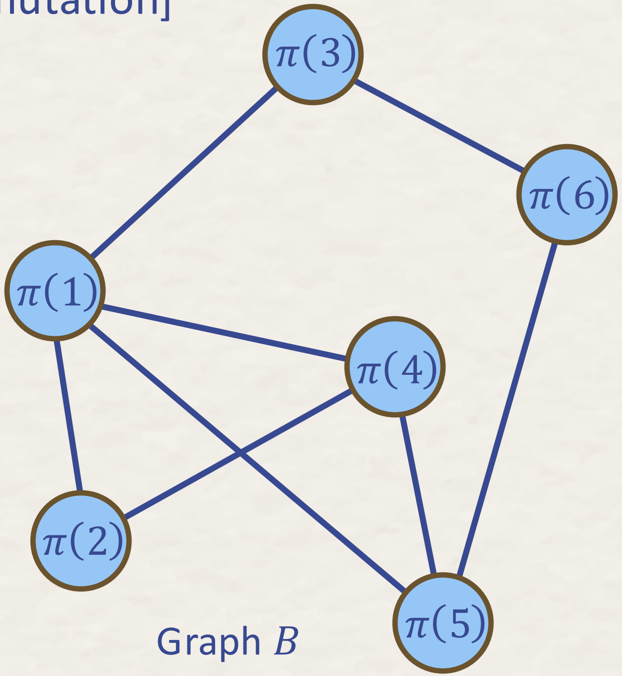
Edge overlap is maximized

$$\max \sum_{i,j=1}^{n} A_{ij} B_{\pi(i)\pi(j)}$$

- Define Permutation $\Pi \in \mathcal{P}$ with $\Pi_{i\pi(i)} = 1$

$$\max_{\Pi \in \mathcal{P}} \langle A, \Pi B \Pi^T \rangle = \min_{\Pi \in \mathcal{P}} ||A\Pi - \Pi B||_F^2$$

Reformulated with a convex objective



Graph B

We still have binary variables $\Rightarrow$ NP-hard to even approximate

Adding Random Structure

- We want A, B to be **random** and **correlated**
- Sample A as a Gaussian Orthogonal Ensemble (GOE) matrix
- $A_{ij} = A_{ji} \sim N\left(0, \frac{1}{n}\right)$

Symmetric matrix with Gaussian entries [Weighted Adjacency Matrix]

- Define B as follows:

$$A \xrightarrow[Z \sim \mathrm{GOE}]{\text{Add Noise}} A + \sigma Z \xrightarrow{\text{Permute}} \Pi^\star (A + \sigma Z) {\Pi^\star}^T = B$$

$\sigma = 0$ is the graph isomorphism problem

$\sigma > 0$ adds noise: A, B are correlated

Graph Alignment with Random Structure

Input

$A, Z \sim \mathrm{GOE}$

$\mathrm{B} = \Pi^\star(A + \sigma Z)\Pi^{\star T}$

Succeeds for $\sigma^2 \leq O(n/\log n)$

[Ganassali 2022], [Wu Xu Yu 2022]

Solve QAP

$\max_{\Pi \in \mathcal{P}} ||A\Pi - \Pi B||_F^2$

Infer $\Pi^\star$

We cannot solve QAP though!

Relax Π by a doubly-stochastic matrix

Birkhoff Relaxation

$\max_{X \in \mathcal{B}} ||AX - XB||_F^2$

Is $X^\star \approx \Pi^\star$?

?

Theorem:

0 — n^{-1} — $n^{-0.5}$ — σ

$||X^\star - \Pi^\star||^2 = o(n)$

???

$||X^\star - \Pi^\star||^2 = \Omega(n)$