Smooth Quadratic Prediction Markets

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Prediction Markets

Core Question: can we design a prediction market which guarantees a better worst-case loss than the common baseline while preserving axiomatic guarantees?

Answer: yes, we propose a new prediction market which preserves most axioms while having a better worst-case loss.

Automated Market Makers

An automated market maker for AD securities, with initial state $q_0 \in \mathbb{R}^d$, operates as follows.

1. A trader can request any bundle of securities $r_t \in \mathbb{R}^d$.
2. The trader pays the market maker some amount Pay$(q_t, r_t) \in \mathbb{R}$ in cash.
3. The market state updates to $q_{t+1} = q_t + r_t$.

After an outcome of the form $Y = y_i$ occurs, for each round t, the trader responsible for the trade r_t is paid $(r_t)_i$ in cash, i.e. the number of shares purchased in outcome y_i. The market payout for the bundle r_t and the outcome $Y = y$ is expressed via $\langle r_t, \rho(y) \rangle$ where $\rho : \mathcal{Y} \to \delta_y$.

Price-Plus Fee Market Construction

A Price-Plus-Fee Market is an automated market maker of the following form. Let $C : \mathbb{R}^d \to \mathbb{R}$ be **CIIP** (convex, increasing, one-invariant, probability mapping). Then

$$\text{Pay}(q_t, r_t) = \langle \nabla C(q_t), r_t \rangle + \text{Fee}(q_t, r_t)$$

We note that in this case, $\text{InstPrice}(q_t) = \nabla C(q_t)$.

Desired Market Axioms

A1 (Existence of Instantaneous Price): C is continuous and differentiable everywhere on $\mathbb{R}^d$.

A2 (Information Incorporation): for any $q, r \in \mathbb{R}^d$,Pay$(q + r, r) \geq$ Pay(q, r).

A3 (No Arbitrage): For all $q, r \in \mathbb{R}^d$, $\exists\ y \in \mathcal{Y}$ such that Pay$(q, r) \geq \langle r, \rho(y) \rangle$.

A4 (Expressiveness): For any $p \in \Delta_d$ and $\epsilon > 0$, $\exists\ q \in \mathbb{R}^d$ such that $||$InstPrice$(q) - p|| < \epsilon$.

A5 (Incentive Compatibility): Assume that the market is at state q_t and that the agent has a belief $\mu \in \Delta_d$. To maximize expected return

$$\arg\max_{r_t \in \mathbb{R}^d} \quad \underbrace{\langle \mu, r_t \rangle}_{\text{Expected Payout}} \quad - \quad \underbrace{\text{Pay}(q_t, r_t)}_{\text{Payment to Market}}$$

the agent will purchase a bundle r_t such that for $q_{t+1} = q_t + r_t$ it holds that InstPrice$(q_{t+1}) = \mu$.

Duality-Based CFMM

In the literature, the Duality-based CFMM has been the most common framework for a prediction market for over the last decade and serves as our baseline.

$$\text{Pay}_D(q_t, r_t) = \langle \nabla C(q_t), r_t \rangle + D_C(q_{t+1}, q_t)$$

- DCFMM satisfies Axioms 1-5
- The DCFMM has a worst cost loss no more than $\sup_{p \in \rho(\mathscr{Y})} C^*(p) - \min_{p \in \Delta_d} C^*(p)$

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Assume that C is L-smooth w.r.t. $\|\cdot\|$, $\quad \text{Pay}_L(q_t, r_t) = \langle \nabla C(q_t), r_t \rangle + \frac{L}{2}\|r_t\|^2$

Theorem 1

- The Smooth Quadratic Prediction Market satisfies Axioms 1-4.
- The Smooth Quadratic Prediction Market has a better worst-case loss than the DCFMM.

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A6 (Incremental Incentive Compatibility): Assume the market is at state q_t and that a sequence of agents with the same belief $\mu \in \Delta_d$ purchases bundles r_t relative to maximizing their expected payout

$$\arg\max_{r_t \in \mathbb{R}^d} \underbrace{\langle \mu, r_t \rangle}_{\text{Expected Payout}} - \underbrace{\text{Pay}(q_t, r_t)}_{\text{Payment to Market}}.$$

Then $\lim_{t \to \infty} \nabla C(q_t) = \mu$.

Theorem 2

- The Smooth Quadratic Prediction Market satisfies Axiom 6.
- The proof is facilitated by showing the agents behavior reduces to ***non-Euclidean steepest descent***.