System-Embedded Diffusion Bridge Models

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Inverse problems

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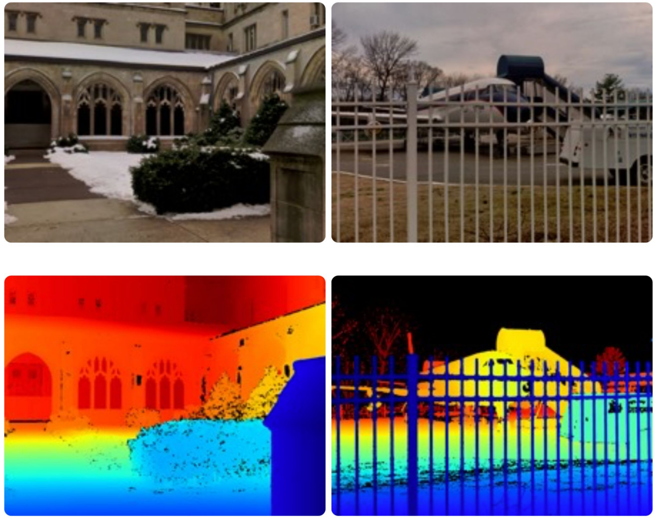
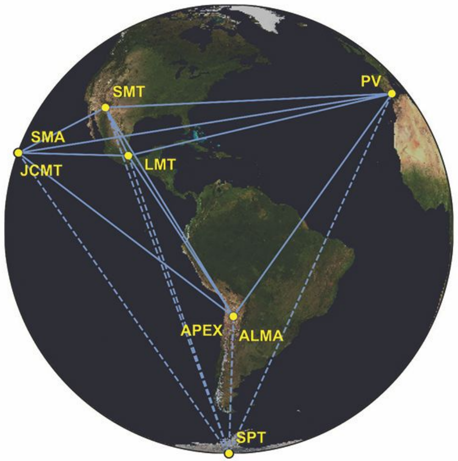
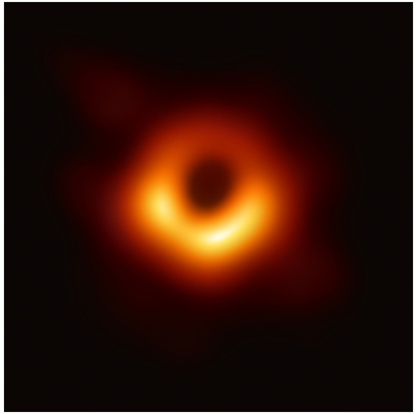
Depth — Scene







Interferometric Data — Black Hole Reconstruction

Day — Night

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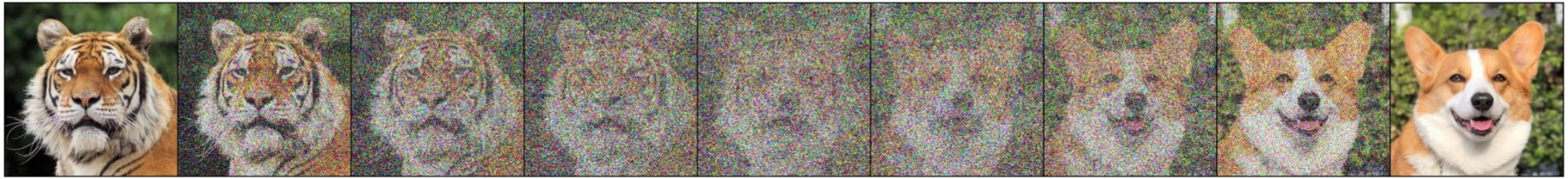
Diffusion bridges

Diffusion bridges

$$(\mathbf{x}, \mathbf{y}) \sim p(\mathbf{x}, \mathbf{y})$$

$$d\boldsymbol{x}_t = \left[\boldsymbol{f}(\boldsymbol{x}_t, t) + g^2(t)\boldsymbol{h}(\boldsymbol{x}_t, t, y, T)\right]dt + g(t)d\boldsymbol{w}_t$$



$$d\boldsymbol{x}_t = \left[\boldsymbol{f}(\boldsymbol{x}_t, t) - g^2(t)\left(\tfrac{1}{2}\boldsymbol{s}(\boldsymbol{x}_t, t, y, T) - \boldsymbol{h}(\boldsymbol{x}_t, t, y, T)\right)\right]dt$$

$\mathbf{x} \sim p(\mathbf{x})$

$\mathbf{y} \sim p(\mathbf{y})$

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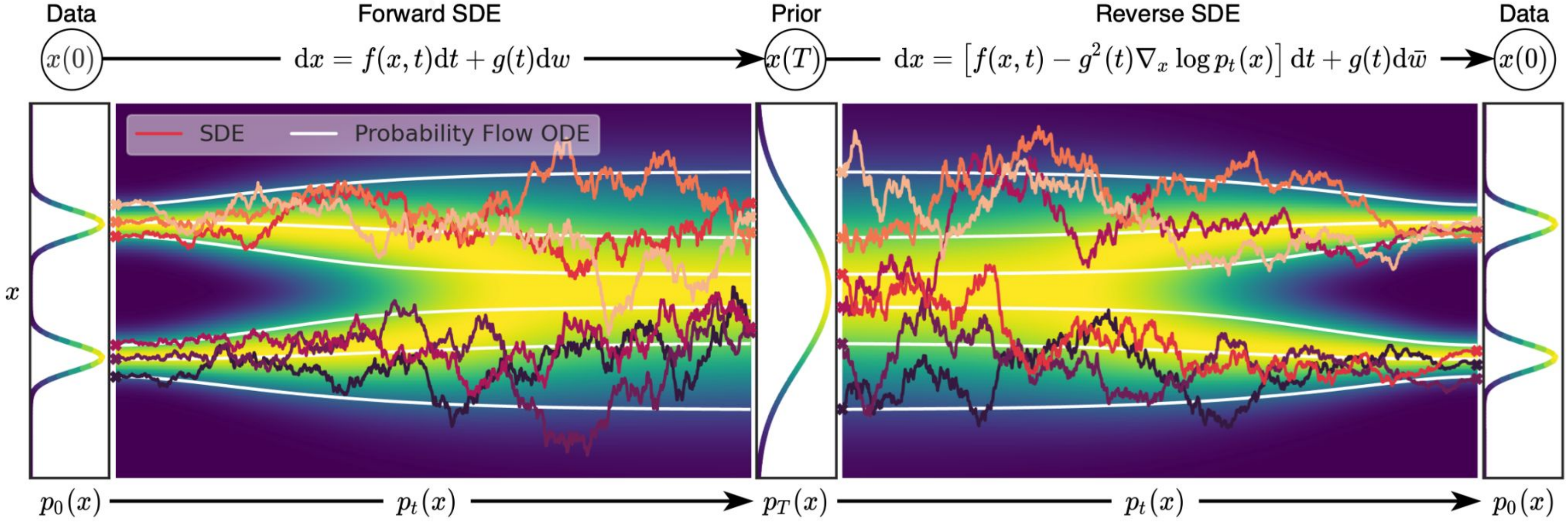
Zhou et al., *Denoising Diffusion Bridge Models*, ICLR 2024

Diffusion models



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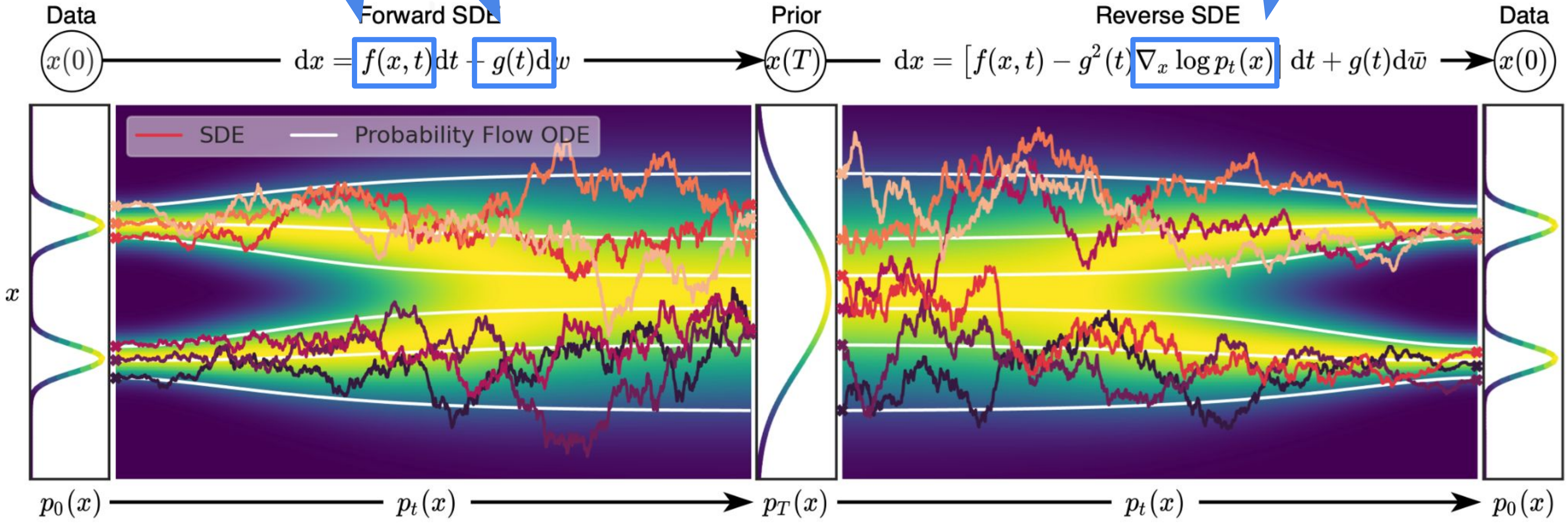
Song et al., *Score-based Generative Modeling Through Stochastic Differential Equations*, ICLR 2021

Diffusion models

Drift coefficient

Diffusion coefficient

Score function



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Song et al., *Score-based Generative Modeling Through Stochastic Differential Equations*, ICLR 2021

What’s missing?

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Forward model

Measurement system

Observable measurement

$$\mathbf{y} = \mathbf{A}(\mathbf{x})$$

Unobservable signal

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Forward model

Linear measurement system

Observable measurement

Noise covariance

$$\mathbf{y} = \mathbf{A}\mathbf{x} + \boldsymbol{\Sigma}^{\frac{1}{2}}\boldsymbol{\epsilon}$$

Unobservable signal

$$\boldsymbol{\epsilon} \sim \mathcal{N}(0, I)$$

Gaussian noise

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Embedding the system into the SDE

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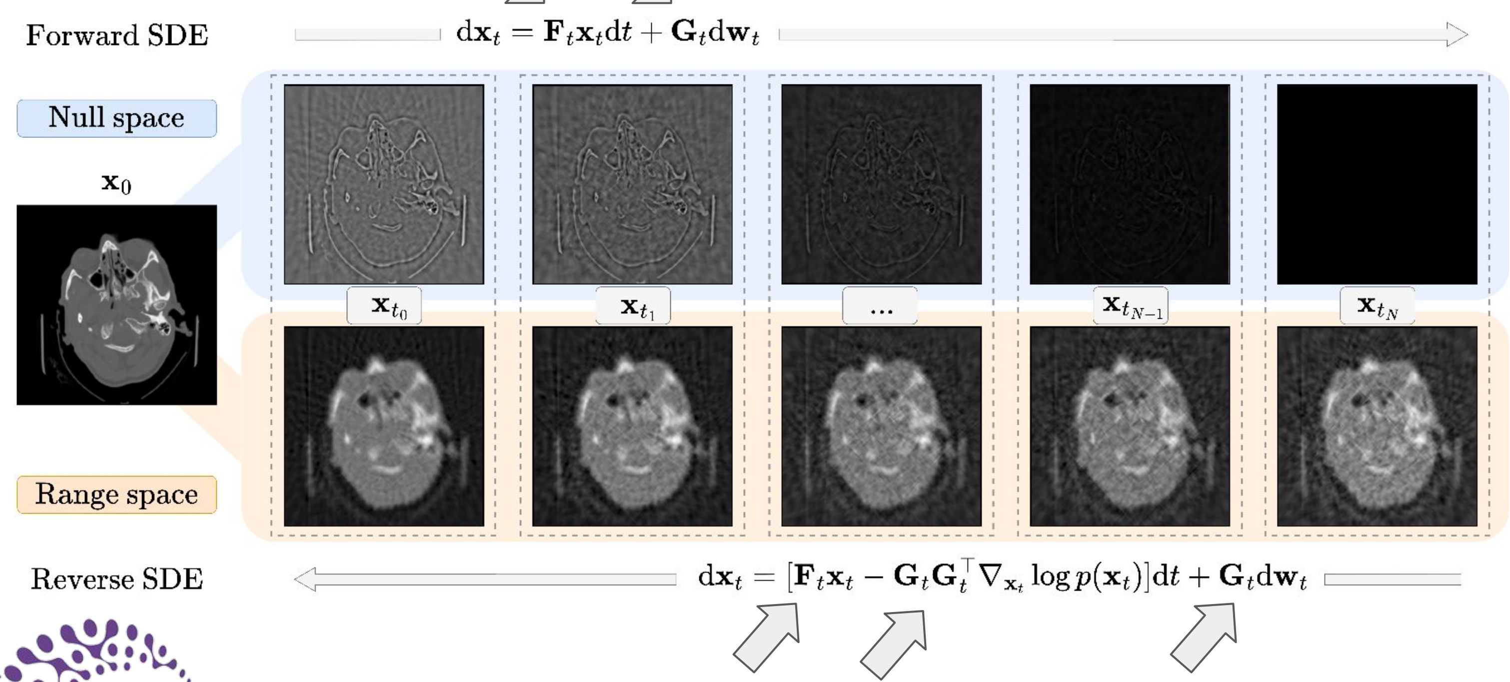
Matrix-valued SDEs

Forward SDE $\mathrm{d}\mathbf{x}_t = \mathbf{F}_t \mathbf{x}_t \mathrm{d}t + \mathbf{G}_t \mathrm{d}\mathbf{w}_t$

Reverse SDE $\mathrm{d}\mathbf{x}_t = [\mathbf{F}_t \mathbf{x}_t - \mathbf{G}_t \mathbf{G}_t^\top \nabla_{\mathbf{x}_t} \log p(\mathbf{x}_t)]\mathrm{d}t + \mathbf{G}_t \mathrm{d}\mathbf{w}_t$

Matrix-valued SDEs



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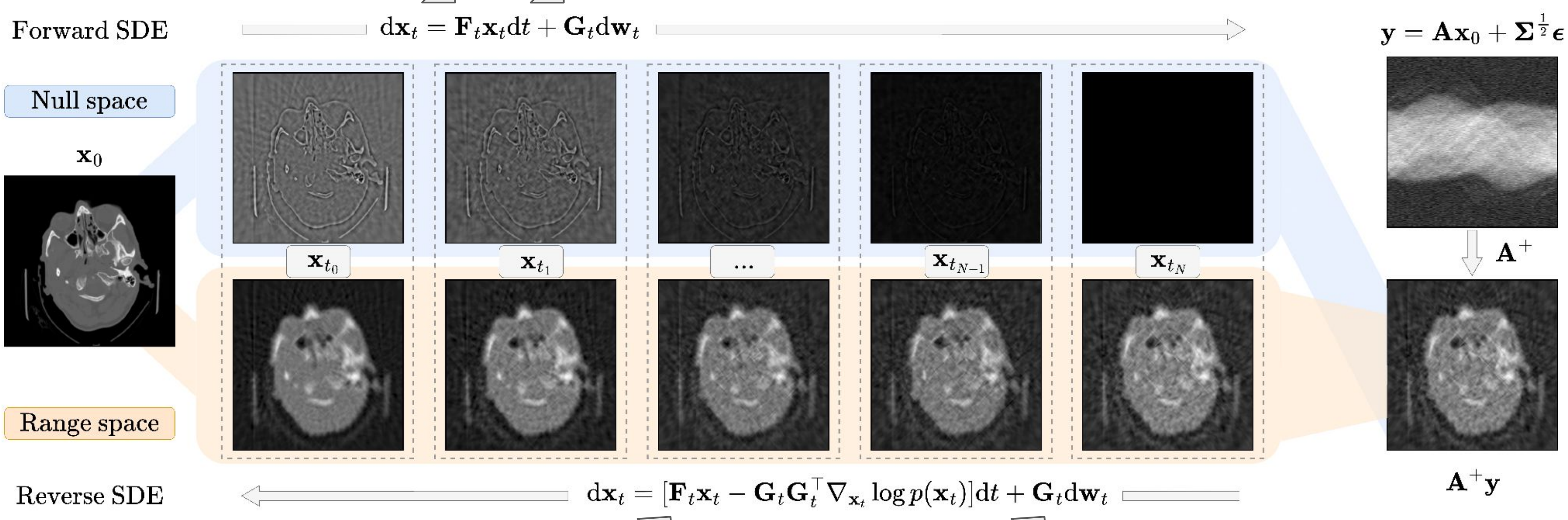
Matrix-valued SDEs



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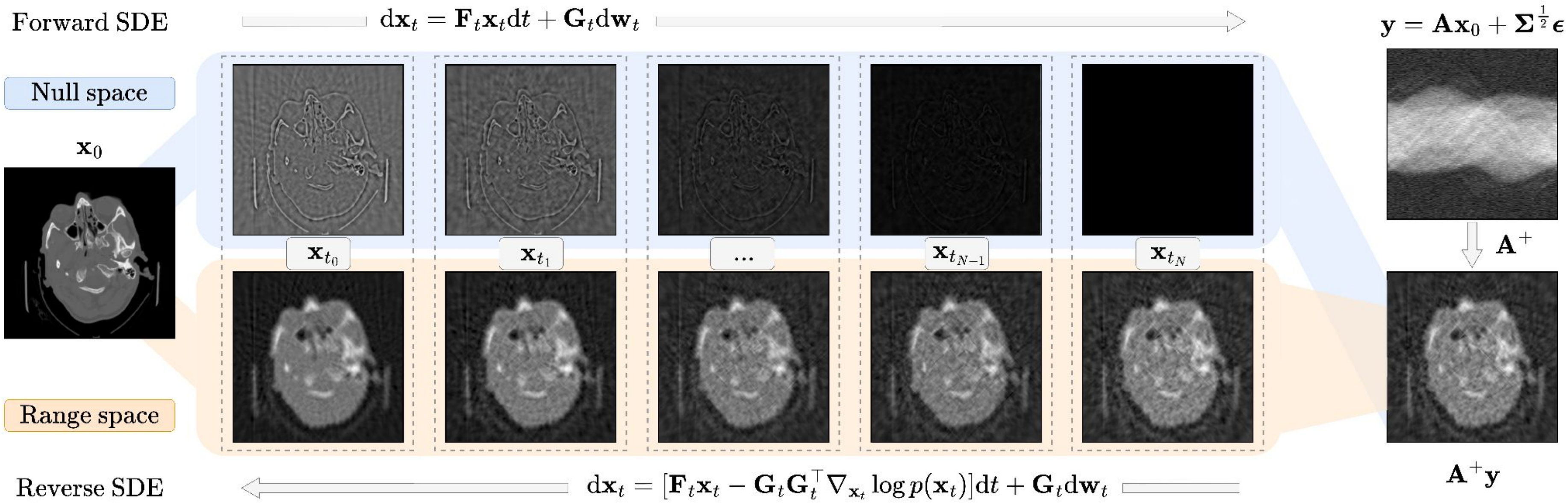
System-Embedded Diffusion Bridge Models (SDB)



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Novel processes

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SDB variants

SDB (SB)

SDB variants

SDB (SB)

SDB (VP)

SDB variants

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Theoretical viewpoint

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Framework generality

Theorem 1 *[Tivnan et al., 2025] Assume that $\mathbf{x}_t \mid \mathbf{x}_0$ evolves according to the linear forward process from eq. (1). Then, $\mathbf{x}_t \mid \mathbf{x}_0 \sim \mathcal{N}(\mathbf{H}_t\mathbf{x}_0, \mathbf{\Sigma}_t)$, where*

$$\mathbf{H}_t = \exp\left(\mathbf{\Omega}_t(\mathbf{F}_t)\right) \approx \exp\left(\int_0^t \mathbf{F}_s ds\right), \mathbf{\Sigma}_t = \mathbf{H}_t \left(\int_0^t \mathbf{H}_s^{-1}\mathbf{G}_s\mathbf{G}_s^\top \mathbf{H}_s^{-1^\top} ds\right)\mathbf{H}_t^\top, \tag{6}$$

where $\mathbf{\Omega}_t$ is the Magnus expansion and the approximation becomes equality if, for all $s, s' \in [0, t]$, $[\mathbf{F}_s, \mathbf{F}_{s'}] = 0$, i.e., $\mathbf{F}_s$ and $\mathbf{F}_{s'}$ commute. Conversely, given $\mathbf{H}_t$ and $\mathbf{\Sigma}_t$, the drift and diffusion coefficients can be obtained via:

$$\mathbf{F}_t = \frac{d\mathbf{H}_t}{dt}\mathbf{H}_t^{-1}, \mathbf{G}_t\mathbf{G}_t^\top = \frac{d\mathbf{\Sigma}_t}{dt} - \mathbf{F}_t\mathbf{\Sigma}_t - \mathbf{\Sigma}_t\mathbf{F}_t^\top. \tag{7}$$

$$\mathbf{H}_t = \mathbf{A}^+\mathbf{A} + \alpha_t(\mathbf{I} - \mathbf{A}^+\mathbf{A}), \tag{8a}$$

$$\mathbf{\Sigma}_t = \gamma_t\mathbf{A}^+\mathbf{\Sigma}\mathbf{A}^{+^\top} + \beta_t(\mathbf{I} - \mathbf{A}^+\mathbf{A}). \tag{8b}$$

$$\mathbf{F}_t = \frac{\mathrm{d}}{\mathrm{d}t}\log\alpha_t(\mathbf{I} - \mathbf{A}^+\mathbf{A}), \tag{10a}$$

$$\mathbf{G}_t\mathbf{G}_t^\top = \frac{\mathrm{d}\gamma_t}{dt}\mathbf{A}^+\mathbf{\Sigma}\mathbf{A}^{+^\top} + \left(\frac{\mathrm{d}\beta_t}{\mathrm{d}t} - 2\beta_t\frac{\mathrm{d}}{\mathrm{d}t}\log\alpha_t\right)(\mathbf{I} - \mathbf{A}^+\mathbf{A}). \tag{10b}$$

Optimal transport

Theorem 2 *For the linear SDE defined in eq. (1) with* $\mathbf{F}_t$ *and* $\mathbf{G}_t$ *given by eqs. (10a) and (10b) respectively, let* $\alpha_t = \frac{\bar{\sigma}_t^2}{\bar{\sigma}_t^2+\sigma_t^2}, \beta_t = \frac{\sigma_t^2\bar{\sigma}_t^2}{\bar{\sigma}_t^2+\sigma_t^2}$ *for* $\sigma_t^2 = \int_0^t g^2(\tau)d\tau, \bar{\sigma}_t^2 = \int_t^1 g^2(\tau)d\tau$ *for some non-negative function* $g(t)$*. Assuming that* $g(t) \to 0$ *uniformly for all* t*, the null space part of this SDE reduces to an OT-ODE:*

$$d(\mathbf{I} - \mathbf{A}^+\mathbf{A})\mathbf{x}_t = \mathbf{v}_t(\mathbf{x}_t \mid \mathbf{x}_0)dt, \tag{11}$$

where $\mathbf{v}_t(\mathbf{x}_t \mid \mathbf{x}_0) = \left(\lim_{g(t)\to 0} \frac{g^2(t)}{\sigma_t^2}\right)(\mathbf{I} - \mathbf{A}^+\mathbf{A})(\mathbf{x}_t - \mathbf{x}_0)$.

SDB (SB)

Principled posterior sampling

Theorem 3 *Let the forward measurement model be* $p(\mathbf{y}|\mathbf{x}) = \mathcal{N}(\mathbf{A}\mathbf{x}, \boldsymbol{\Sigma})$. *Then, under the SDB dynamics defined by eq. (1), eq. (7), and eq. (8), with time-dependent scalar coefficients* α_t, β_t, *and* γ_t *satisfying*

$$\lim_{t\to 1} \gamma_t = 1, \quad \lim_{t\to 1} \frac{\alpha_t^2}{\beta_t} = 0,$$

the corresponding reverse-time SDE generates asymptotically exact samples from the Bayesian posterior distribution $p(\mathbf{x}|\mathbf{y})$.

$$p(\mathbf{x} \mid \mathbf{y})$$

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Empirical results

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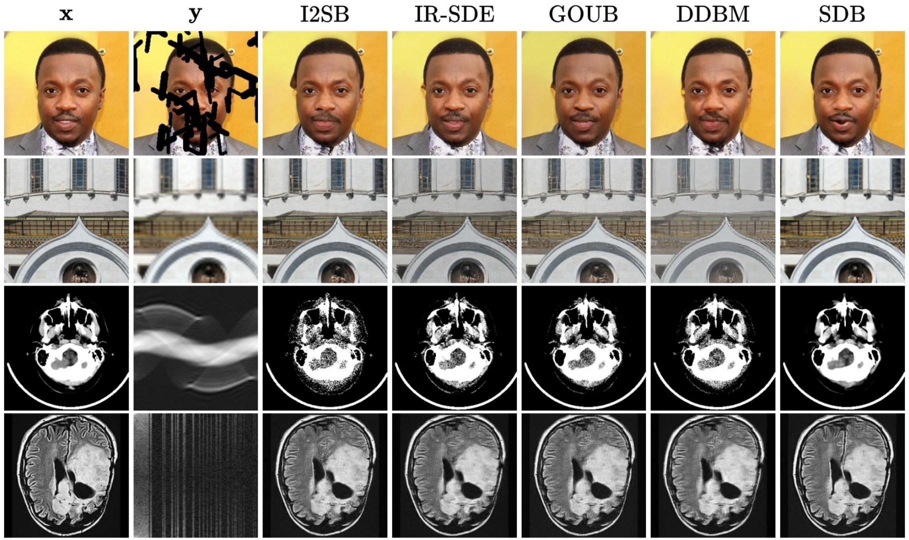
New state-of-the-art



Method	Inpainting - CelebA-HQ				Superresolution - DIV2K				CT Reconstruction - RSNA				MRI Reconstruction - Br35H			
	FID ↓	LPIPS ↓	PSNR ↑	SSIM ↑	FID ↓	LPIPS ↓	PSNR ↑	SSIM ↑	FID ↓	LPIPS ↓	PSNR ↑	SSIM ↑	FID ↓	LPIPS ↓	PSNR ↑	SSIM ↑
Unsupervised																
DPS	12.3	0.213	18.59	0.684	101.2	0.295	19.04	0.492	29.25	0.160	32.870	0.709	40.73	0.187	21.336	0.549
ΠGDM	10.2	0.146	19.02	0.762	97.09	0.311	18.84	0.530	32.29	0.165	31.272	0.757	41.14	0.211	19.830	0.665
DDNM	10.7	0.112	18.90	0.791	99.21	0.302	17.13	0.513	30.41	0.238	27.226	0.783	42.23	0.196	18.122	0.636
Supervised																
I2SB	5.56	0.047	27.41	0.889	83.73	**0.176**	25.21	0.686	24.81	0.107	41.886	0.924	31.54	0.065	28.750	0.849
IR-SDE	4.68	0.031	29.92	0.912	96.22	0.185	23.51	0.603	18.88	0.028	43.438	0.964	30.14	0.065	28.878	0.871
GOUB	4.69	0.031	29.89	0.912	98.89	0.178	24.39	0.649	19.90	0.024	43.878	0.967	30.63	0.058	28.590	0.863
DDBM	6.03	0.047	28.15	0.906	90.16	0.233	25.79	0.720	23.36	0.040	44.415	0.964	32.42	0.074	28.971	0.872
Ours																
SDB (VP)	4.90	**0.030**	30.51	0.914	87.08	0.228	25.91	**0.724**	15.43	0.019	46.365	0.981	32.88	0.068	29.255	0.881
SDB (VE)	5.97	0.042	**32.12**	**0.944**	94.73	0.226	25.90	0.718	**14.17**	0.020	46.325	0.981	33.90	0.083	29.098	0.876
SDB (SB)	**4.63**	0.031	30.40	0.930	**81.56**	0.226	**26.10**	**0.724**	15.02	**0.018**	**46.672**	**0.982**	**29.85**	**0.053**	**29.812**	**0.893**

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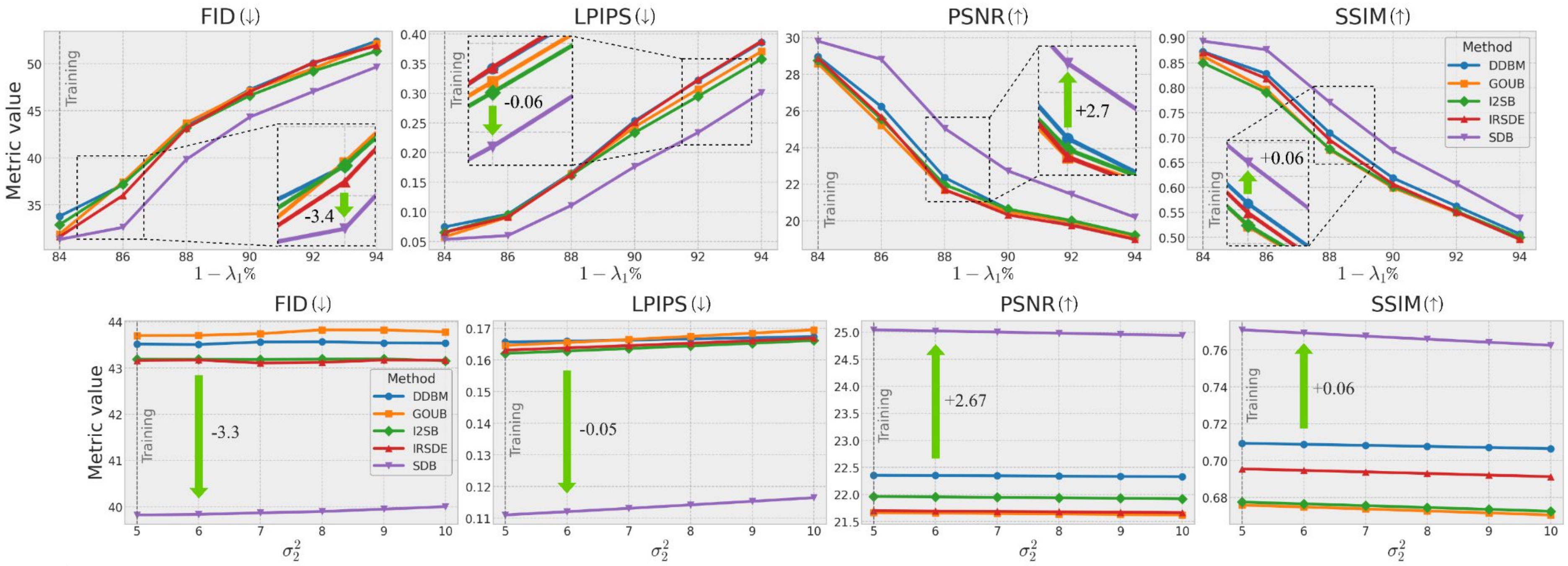
Misspecified model



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Misspecified operator and noise

Method	Kernel 32×32				JPEG 30%			
	FID↓	LPIPS↓	PSNR↑	SSIM↑	FID↓	LPIPS↓	PSNR↑	SSIM↑
IRSDE	130.47	0.684	17.85	0.381	99.43	**0.283**	24.81	0.651
I2SB	131.33	**0.681**	17.84	0.381	93.62	0.322	25.12	0.668
DDBM	131.55	0.700	17.83	0.380	92.39	0.373	25.19	0.685
GOUB	134.66	0.690	17.85	0.370	101.08	0.291	24.91	0.662
SDB (SB)	**129.87**	0.682	**17.86**	**0.390**	**90.26**	0.380	**25.41**	**0.692**

Method	$\mathbf{I}_0 = 10000$				$\mathbf{I}_0 = 5000$				$\mathbf{I}_0 = 1000$			
	FID↓	LPIPS↓	PSNR↑	SSIM↑	FID↓	LPIPS↓	PSNR↑	SSIM↑	FID↓	LPIPS↓	PSNR↑	SSIM↑
GOUB	17.78	0.0230	44.85	0.9717	17.83	0.0240	44.59	0.9705	20.52	0.0491	42.79	0.9588
IRSDE	17.29	0.0256	44.17	0.9681	17.86	0.0299	43.88	0.9665	21.12	0.0678	41.84	0.9506
DDBM	20.69	0.0402	44.11	0.9622	20.57	0.0406	43.93	0.9613	21.75	0.0575	42.53	0.9525
I2SB	21.91	0.1014	41.68	0.9242	21.82	0.1018	41.54	0.9231	22.43	0.1170	40.44	0.9130
SDB (SB)	**14.03**	**0.0193**	**46.05**	**0.9799**	**14.61**	**0.0203**	**45.70**	**0.9785**	**19.80**	**0.0461**	**43.44**	**0.9657**

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What's next?

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Future work

1. Nonlinear forward models

Future work

1. Nonlinear forward models
2. General operator-specific SDEs

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Future work

1. Nonlinear forward models

2. General operator-specific SDEs

3. Trustworthiness

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Thank you!

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References

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3. Luo et al., *Image Restoration with Mean-Reverting Stochastic Differential Equations*, ICML 2023,
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6. Song et al., *Pseudoinverse-guided diffusion models for inverse problems*, ICLR 2023,
7. Yue et al., *Image Restoration Through Generalized Ornstein-Uhlenbeck Bridge*, ICML 2024,
8. Zhou et al., *Denoising Diffusion Bridge Models*, ICLR 2024

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