

PINNs with Learnable Quadrature



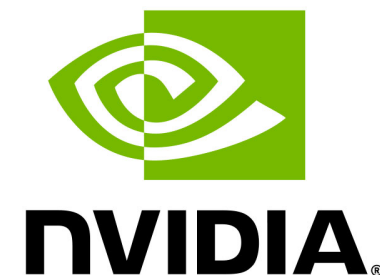
Sourav Pal



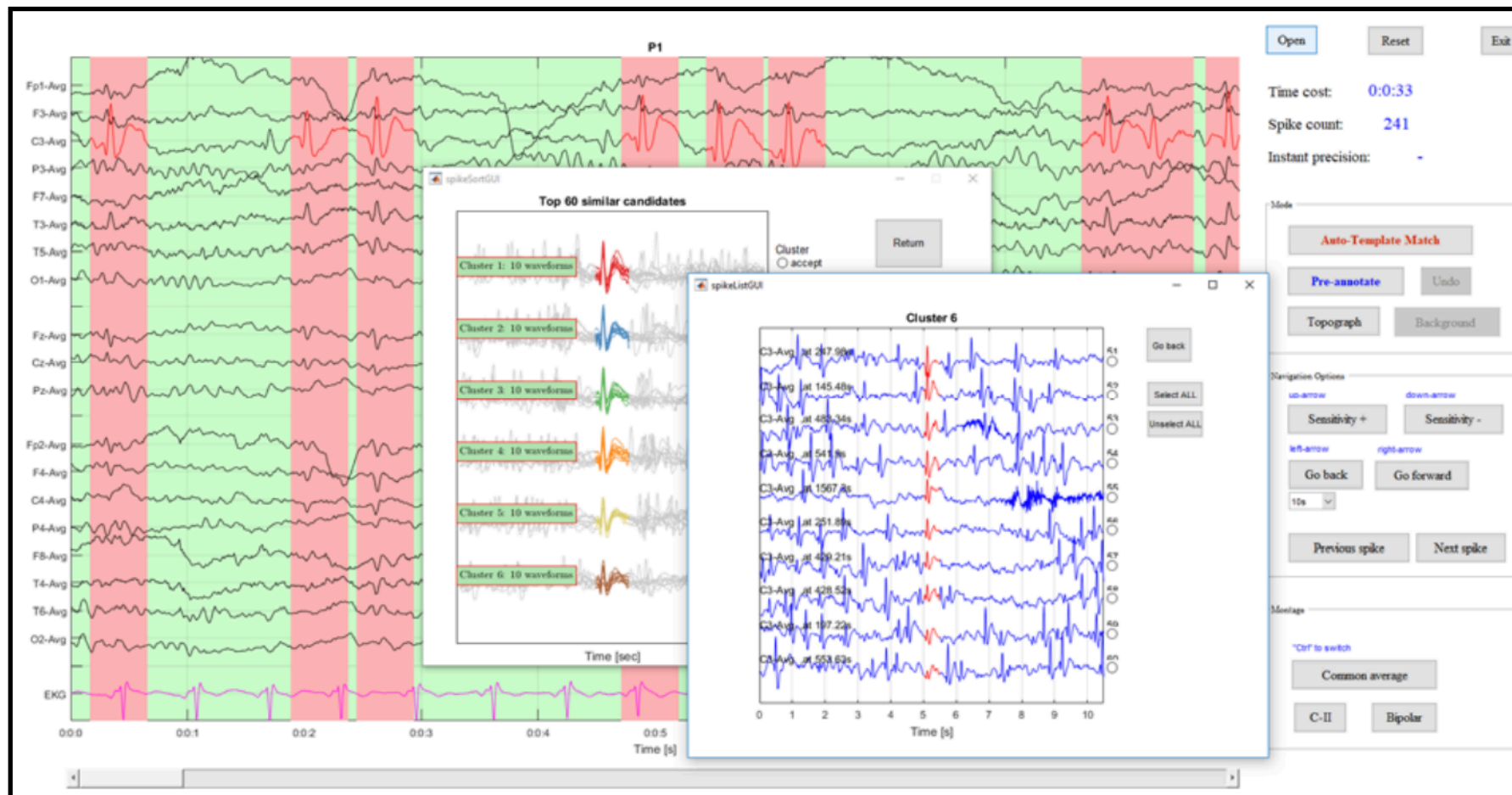
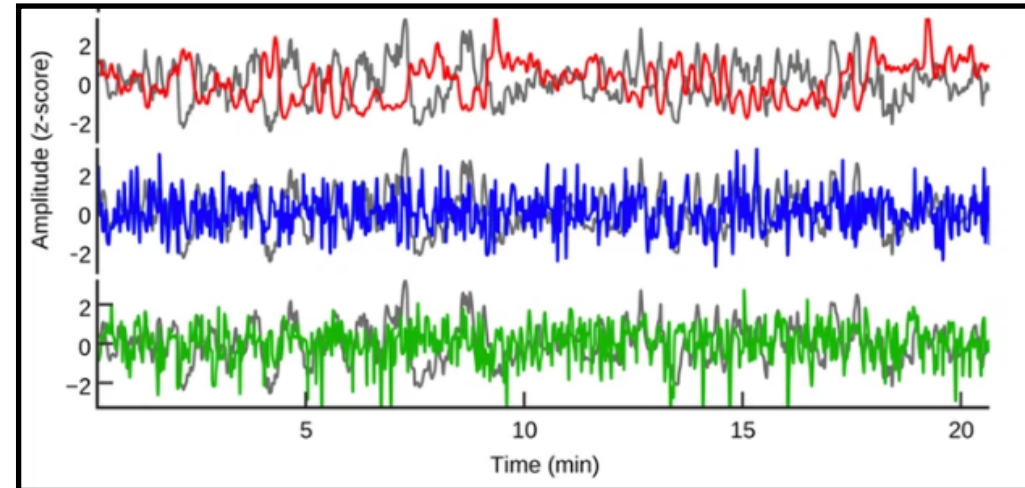
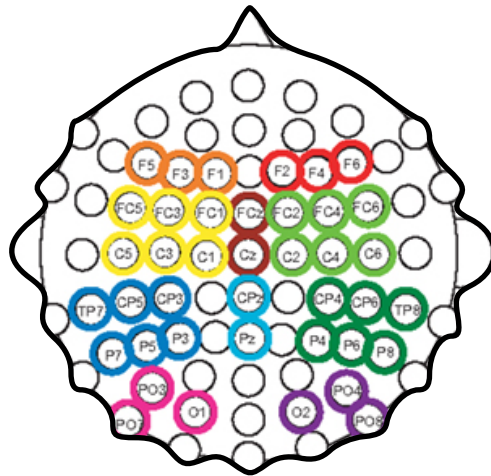
Kamyar Azizzadenesheli



Vikas Singh



Motivation



Physics-informed machine learning

George Em Karniadakis^{1,2}, Ioannis G. Kevrekidis^{3,4}, Lu Lu⁵, Paris Perdikaris⁶,
Sifan Wang⁷ and Liu Yang¹

Differential Equation:

$$\mathcal{N}[u(\mathbf{x}, t)] = 0, \quad \mathbf{x} \in \Omega, t \in [0, T]$$

$$u(\mathbf{x}, 0) = u_0(\mathbf{x}), \quad \mathbf{x} \in \Omega$$

$$u(\mathbf{x}, t) = g(\mathbf{x}, t), \quad \mathbf{x} \in \partial\Omega$$

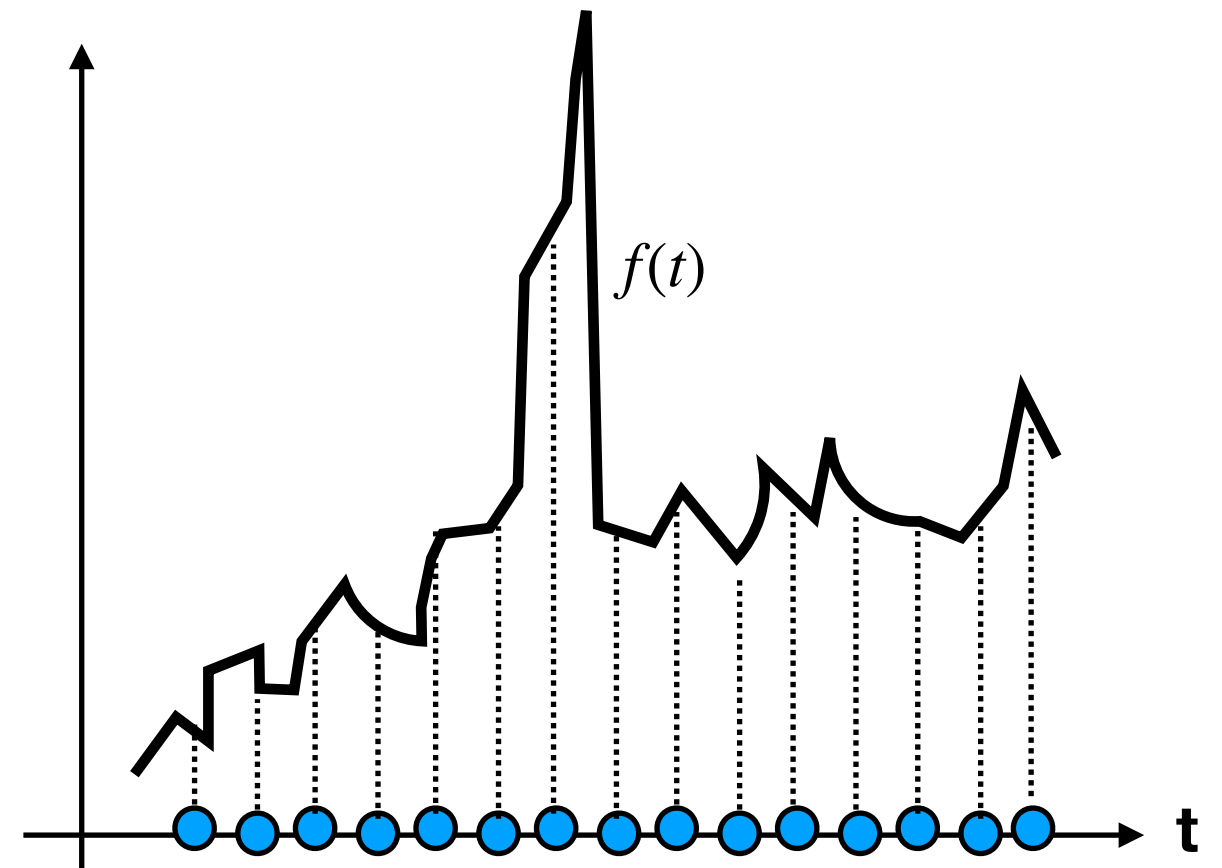
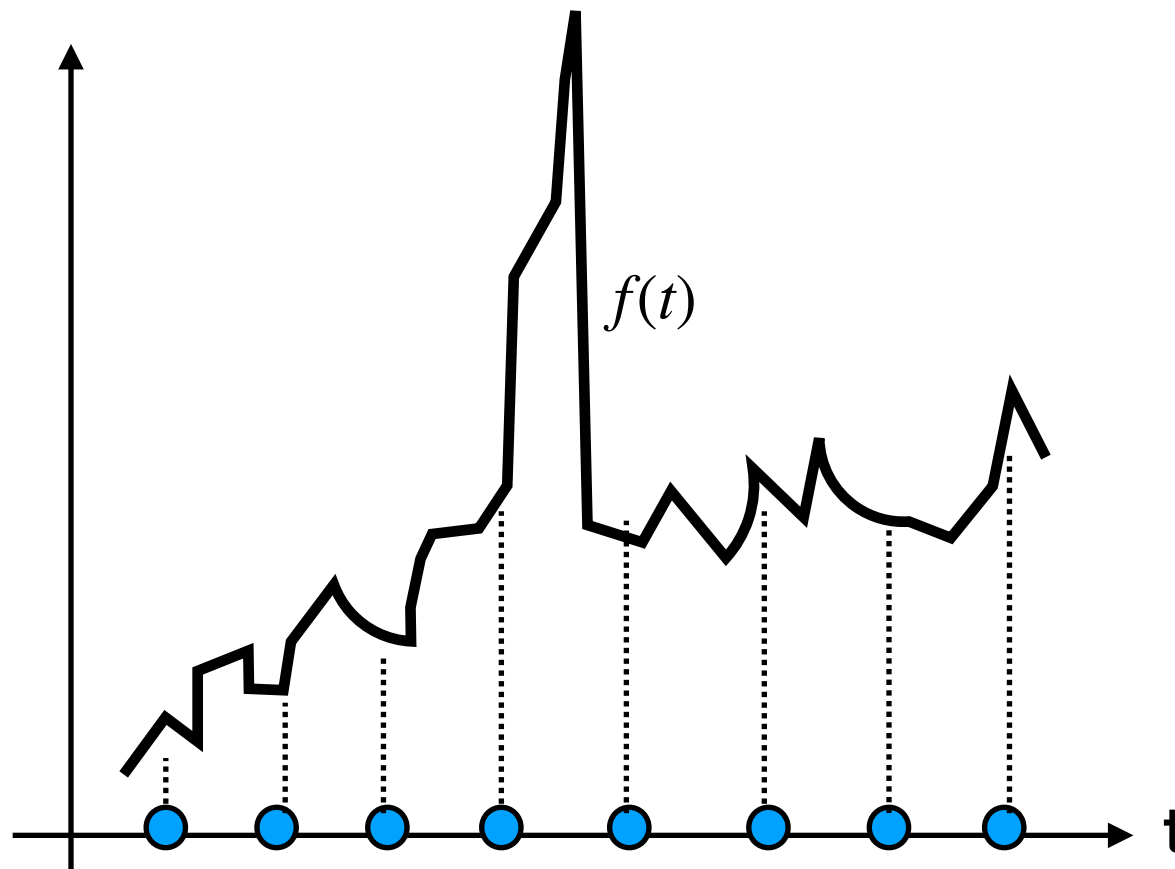
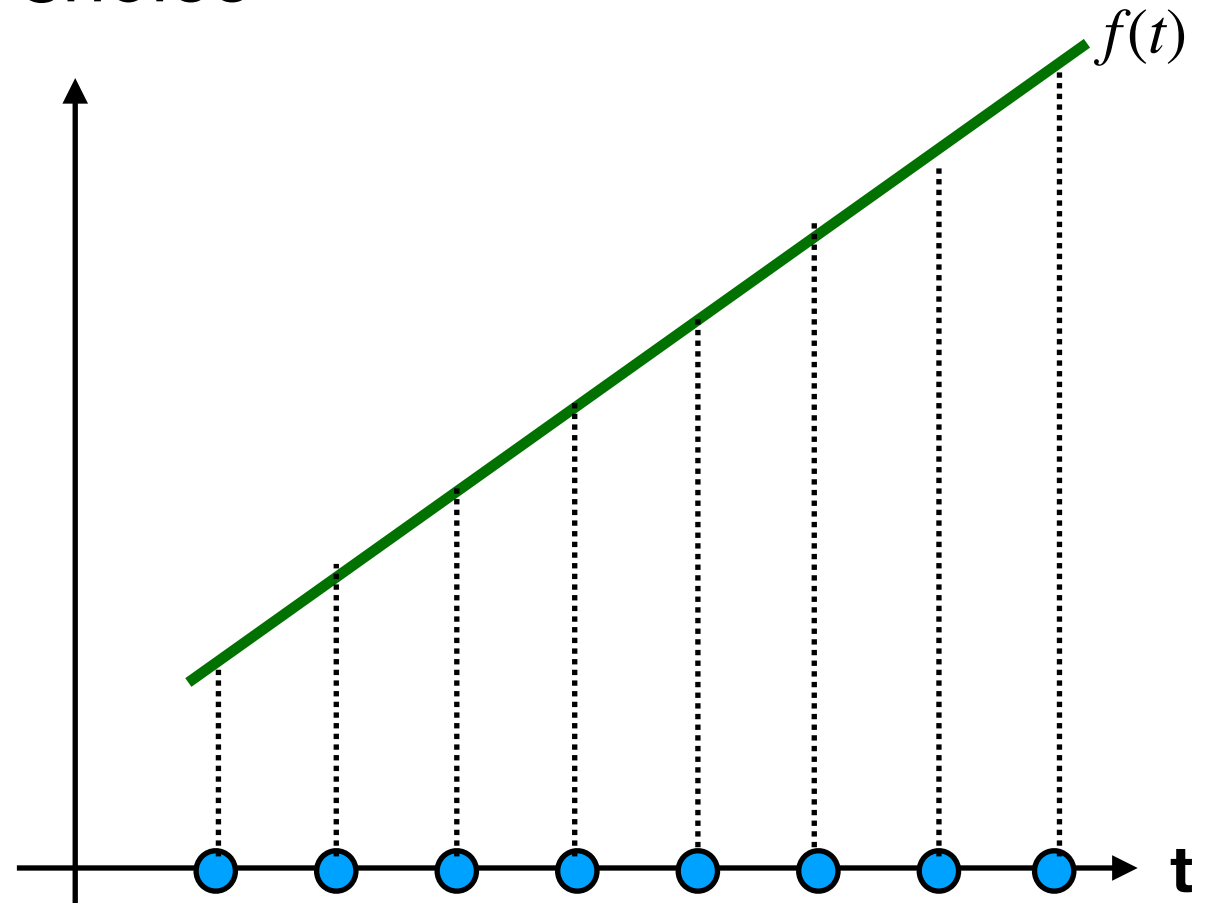
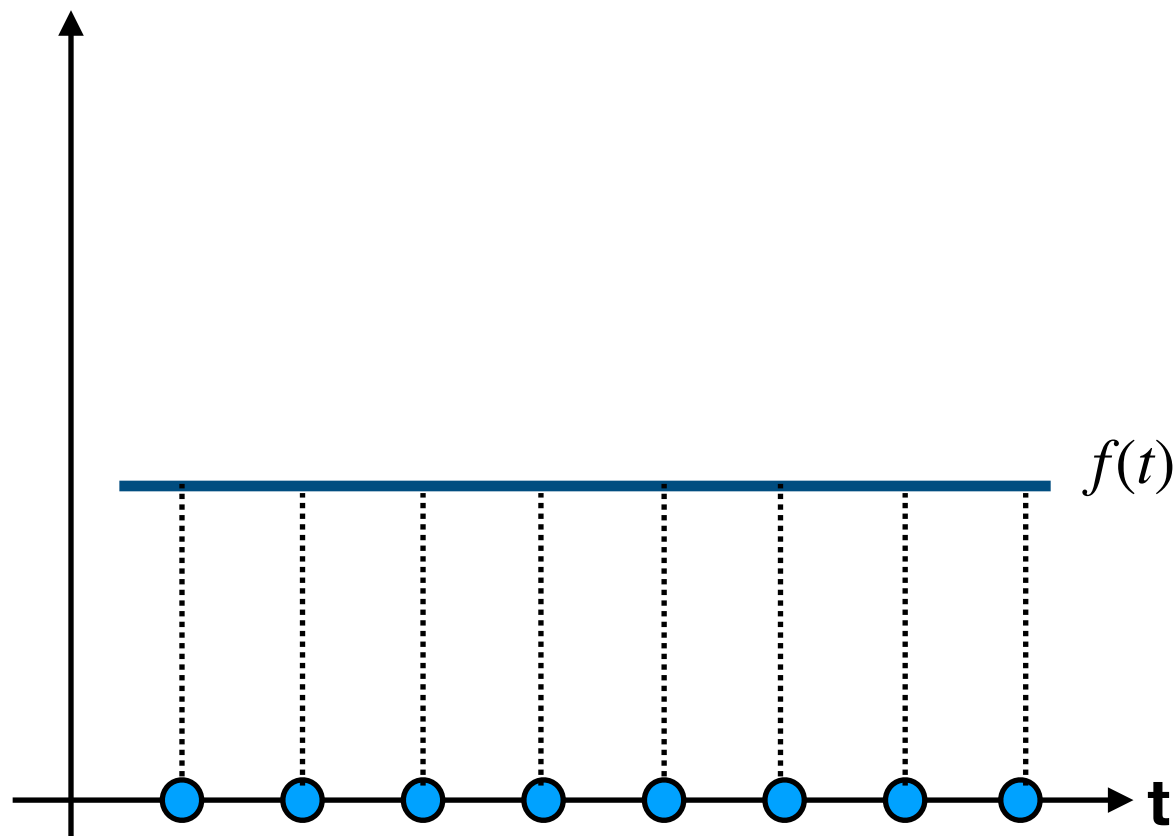
Parameterized Solution:

$$u(\mathbf{x}, t) \approx u_\theta(\mathbf{x}, t)$$

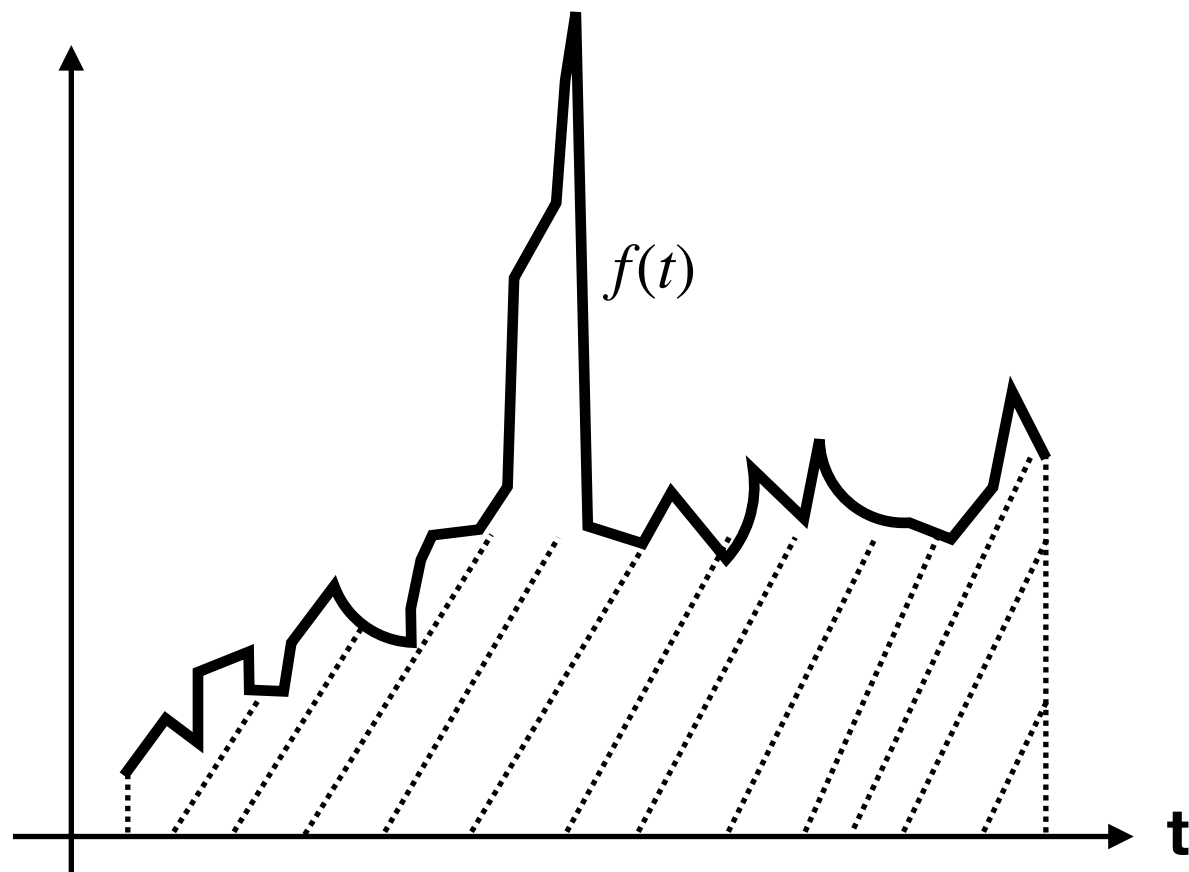
Loss:

$$\mathcal{L}(\theta) = \underbrace{\frac{1}{N_{IC}} \sum_{i=1}^{N_{IC}} \left| u_\theta(\mathbf{x}_i, 0) - u_0(\mathbf{x}_i) \right|^2}_{\text{Initial Condition Loss}} + \underbrace{\frac{1}{N_{BC}} \sum_{j=1}^{N_{BC}} \left| u_\theta(\mathbf{x}_j, t_j) - g(\mathbf{x}_j, t_j) \right|^2}_{\text{Boundary Condition Loss}} + \underbrace{\lambda \frac{1}{N_f} \sum_{k=1}^{N_f} \left| \mathcal{N}[u_\theta(\mathbf{x}_k, t_k)] \right|^2}_{\text{PDE Residual Loss}}$$

Sampling Choice



Computing the integral

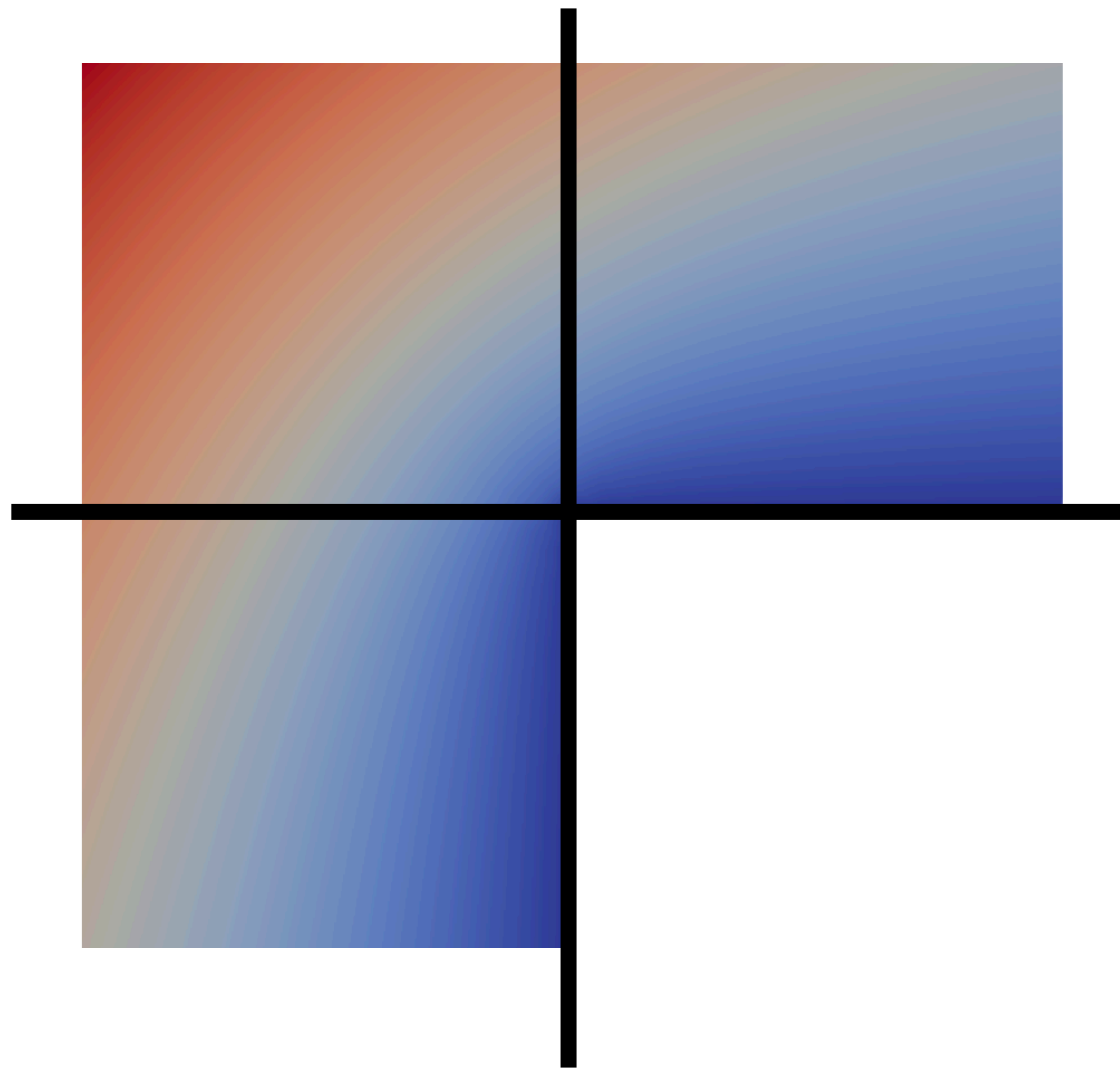


No analytic solution

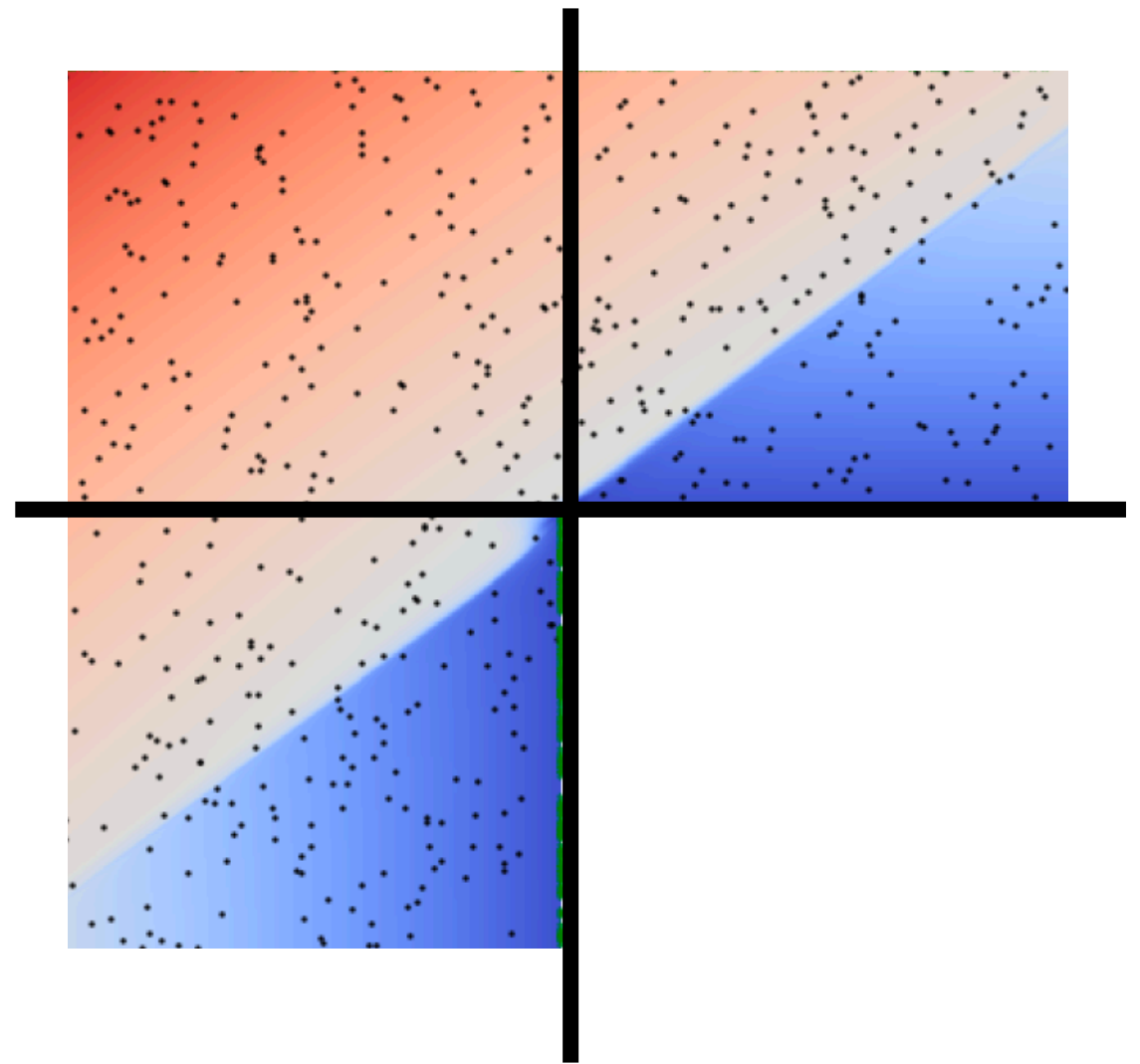
Uniform quadrature

Monte Carlo: High dimensions
Low dimensions

Random Quadrature is Sub-Optimal



True Solution



Random quadrature

Strong & Weak Form

$$\begin{aligned}\mathcal{L}u &= f, & \text{in } \Omega \\ u &= g, & \text{in } \partial\Omega\end{aligned}$$

$$\begin{aligned}-\nabla^2 u(x, y) &= f(x, y), & (x, y) \in \Omega \\ u(x, y) &= 0, & (x, y) \in \partial\Omega\end{aligned}$$

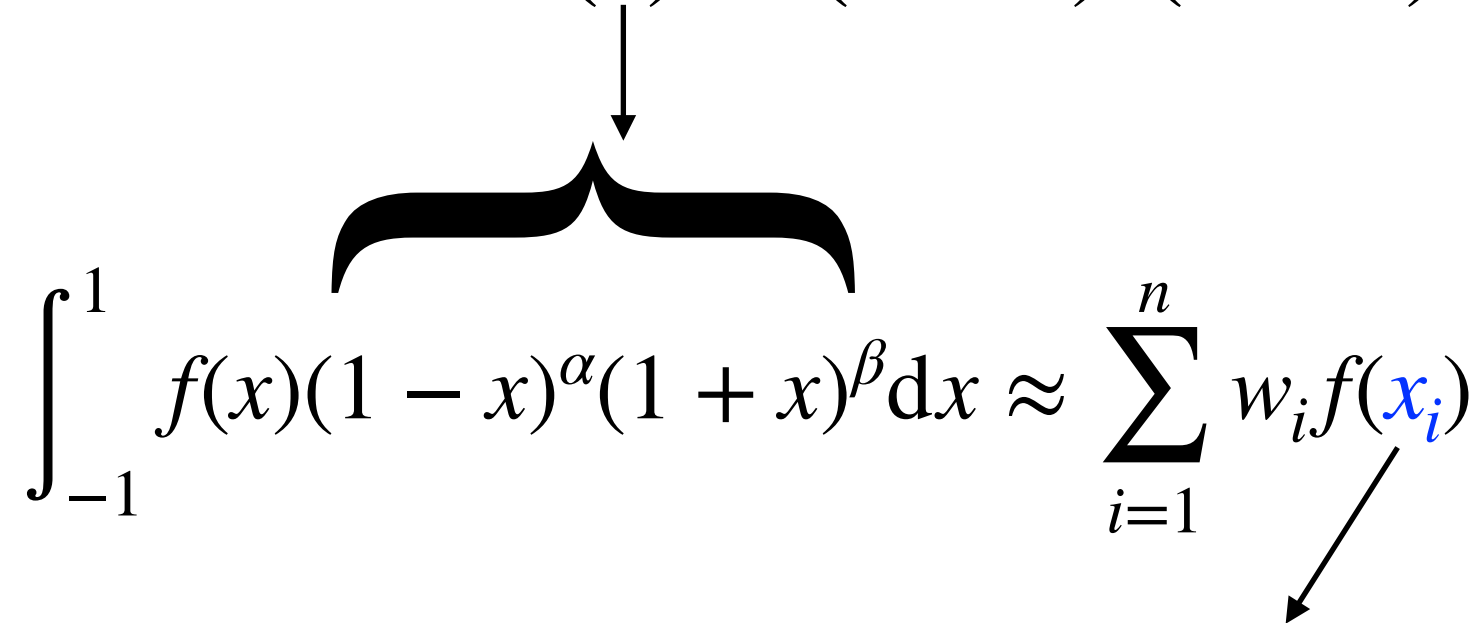
Choice of Collocation Points

$$\iint_{\Omega} -\nabla^2 u \underbrace{v}_{\text{Test function}} \, dx dy = \iint_{\Omega} f \underbrace{v}_{\text{Test function}} \, dx dy$$

Choice of Test functions
Choice of Quadrature Rules

Quadrature Rule

Weight function: $w(x) = (1 - x)^\alpha(1 + x)^\beta$


$$\int_{-1}^1 f(x)(1 - x)^\alpha(1 + x)^\beta dx \approx \sum_{i=1}^n w_i f(x_i)$$

Roots of Orthogonal Polynomial

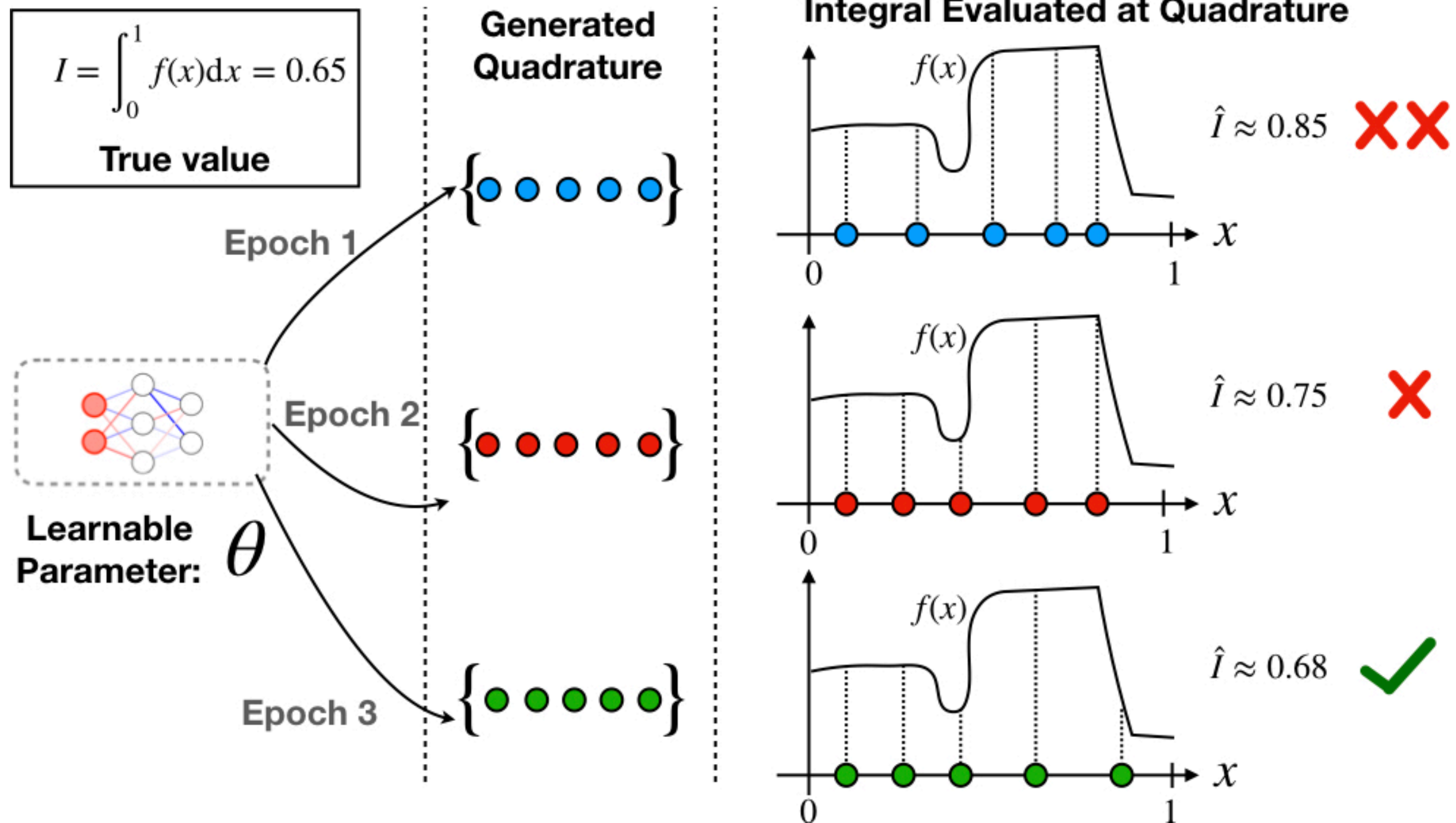
**Generalized Gauss Jacobi
type weight:**

$$w(x) = (1 - x)^\alpha(1 + x)^\beta h(x), \quad x \in [-1, 1]$$

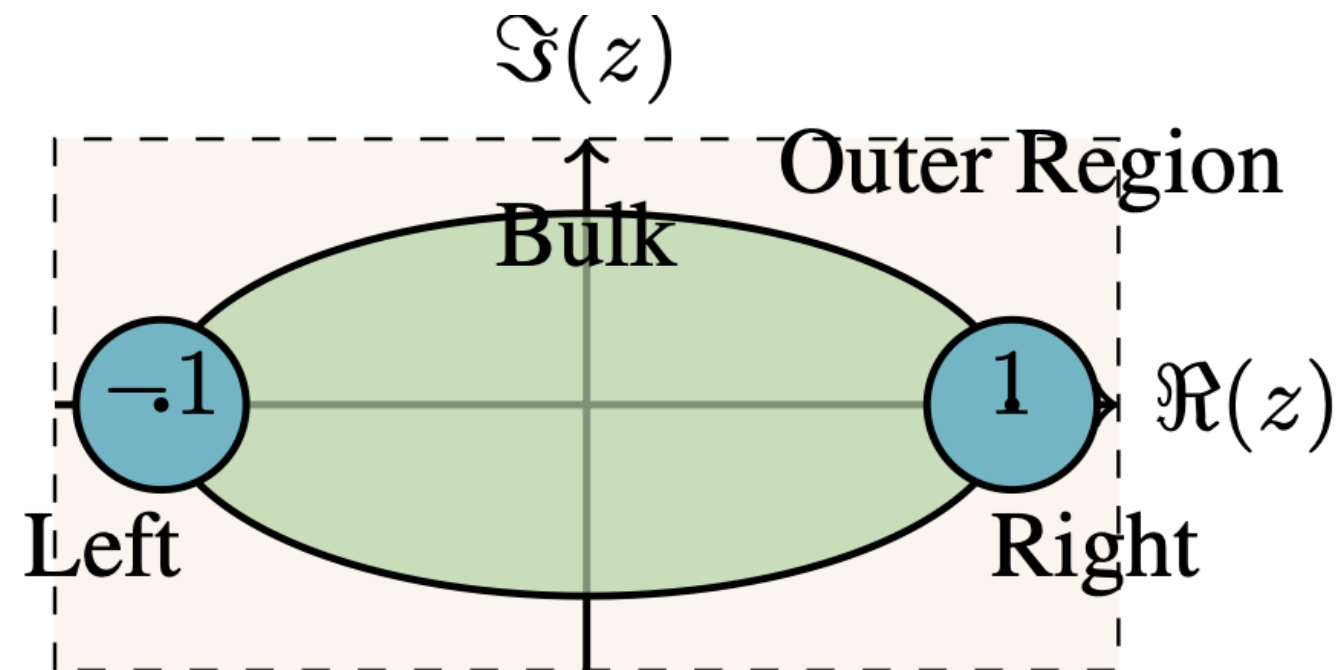
Fast computation of nodes and weights for large n!!

Idea: Jointly learn the test (weight) functions and solution

Learnable Quadrature



Asymptotic Formulae



Value depends on the region of the domain

Bulk Region $x_k \sim t_k + (\text{Other terms})$

$$\frac{w_k}{w(x_k)} \sim \frac{\pi \sqrt{1 - t_k^2}}{\Gamma} + (\text{Other terms})$$

Other Terms depend on α, β, n, h

Parameterization & Regularization

Small MLPs (tanh): Solution Function u_θ

Coefficients in the asymptotic expansion

Weight Function $w(x) = (1-x)^\alpha(1+x)^\beta h_\theta(x)$, $x \in [-1,1]$

$$Loss = \underbrace{L(u_\theta(x_k))}_{\text{PINN Loss}} + \underbrace{\left(\sum_{i=1}^n w_i - \int_{-1}^1 w_\theta(x) dx\right)^2}_{\text{Valid Quadrature}} + \underbrace{\left(\sum_{i=1}^n w_i - 2\right)^2}_{\text{Non-trivial weight}}$$

PINN Loss

Valid Quadrature

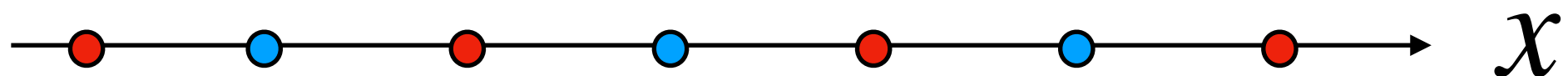
Non-trivial weight

Naive application leads to overfitting

Interlacing of roots!

p_n

p_{n+1}



Metric and Evaluation

$$\text{Relative Error} = \frac{||u_\theta - u||_2}{||u||_2}$$

	PDE # points	Burgers' 2000	Allen-Cahn 1000	Wave 2000
Non-adaptive	Grid	0.12 ± 0.04	0.88 ± 0.06	0.42 ± 0.09
	Random	0.13 ± 0.03	0.32 ± 0.14	0.48 ± 0.07
	LHS	0.18 ± 0.15	0.32 ± 0.04	0.61 ± 0.13
	Halton	0.06 ± 0.02	0.18 ± 0.05	0.46 ± 0.06
	Hammersley	0.07 ± 0.05	0.17 ± 0.05	0.31 ± 0.09
	Sobol	0.08 ± 0.03	0.20 ± 0.10	0.49 ± 0.09
Adaptive	Random-R	1.69 ± 1.67	0.55 ± 0.34	0.72 ± 0.90
	RAR-G	0.12 ± 0.04	0.53 ± 0.19	0.81 ± 0.11
	RAD	0.02 ± 0.00	0.08 ± 0.06	0.09 ± 0.04
	RAR-D	0.03 ± 0.01	0.09 ± 0.03	0.29 ± 0.04
	LearnQuad	0.003 ± 0.002	0.03 ± 0.008	0.005 ± 0.0006

Table 1: LearnQuad has lowest L2 relative error (mean ± standard deviation)

Metric and Evaluation

$$\text{Relative Error} = \frac{||u_\theta - u||_2}{||u||_2}$$

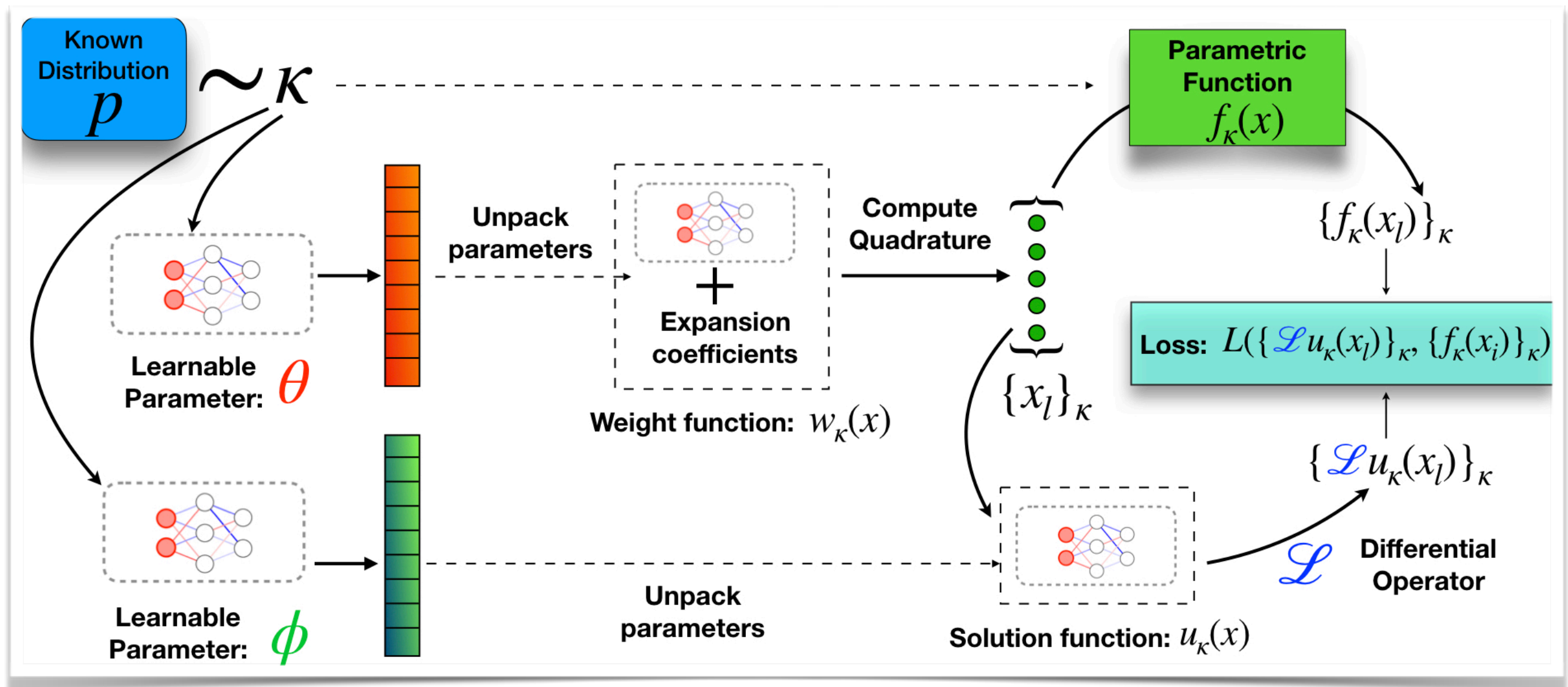
PDE Epochs.	Convection ($\beta = 30$)		Convection ($\beta = 50$)		Allen Cahn
	100k	300k	150k	300k	200k
PINN (fixed)	$107.5 \pm 10.9\%$	$107.5 \pm 10.7\%$	$108.5 \pm 6.38\%$	$108.7 \pm 6.59\%$	$69.4 \pm 4.02\%$
PINN (dynamic)	$2.81 \pm 1.45\%$	$1.35 \pm 0.59\%$	$24.2 \pm 23.2\%$	$56.9 \pm 9.08\%$	$0.77 \pm 0.06\%$
Curr Reg	$63.2 \pm 9.89\%$	$2.65 \pm 1.44\%$	$48.9 \pm 7.44\%$	$31.5 \pm 16.6\%$	–
CPINN (fixed)	$138.8 \pm 11.0\%$	$138.8 \pm 11.0\%$	$106.5 \pm 10.5\%$	$106.5 \pm 10.5\%$	$48.7 \pm 19.6\%$
CPINN (dynamic)	$52.2 \pm 43.6\%$	$23.8 \pm 45.1\%$	$79.0 \pm 5.11\%$	$73.2 \pm 3.6\%$	$1.5 \pm 0.75\%$
RAR-G	$10.5 \pm 5.67\%$	$2.66 \pm 1.41\%$	$65.7 \pm 1.77\%$	$43.1 \pm 28.9\%$	$25.1 \pm 23.2\%$
RAD	$3.35 \pm 2.02\%$	$1.85 \pm 1.90\%$	$66.0 \pm 1.55\%$	$64.1 \pm 11.9\%$	$0.78 \pm 0.05\%$
RAR-D	$67.1 \pm 4.28\%$	$32.0 \pm 25.8\%$	$82.9 \pm 5.96\%$	$75.3 \pm 9.58\%$	$51.6 \pm 0.41\%$
L^∞	$66.6 \pm 2.35\%$	$41.2 \pm 27.9\%$	$76.6 \pm 1.04\%$	$75.8 \pm 1.01\%$	$1.65 \pm 1.36\%$
R3	$1.51 \pm 0.26\%$	$0.78 \pm 0.18\%$	$1.98 \pm 0.72\%$	$2.28 \pm 0.76\%$	$0.83 \pm 0.15\%$
Causal R3	$2.12 \pm 0.67\%$	$0.75 \pm 0.12\%$	$5.99 \pm 5.25\%$	$2.28 \pm 0.76\%$	$0.71 \pm 0.007\%$
LearnQuad	$0.78 \pm 0.002\%$	$0.68 \pm 0.02\%$	$0.79 \pm 0.02\%$	$0.76 \pm 0.01\%$	$0.87 \pm 0.01\%$

Table 2: Relative L_2 error with 1000 collocation points. LearnQuad yields best results in 4 of 5 cases.

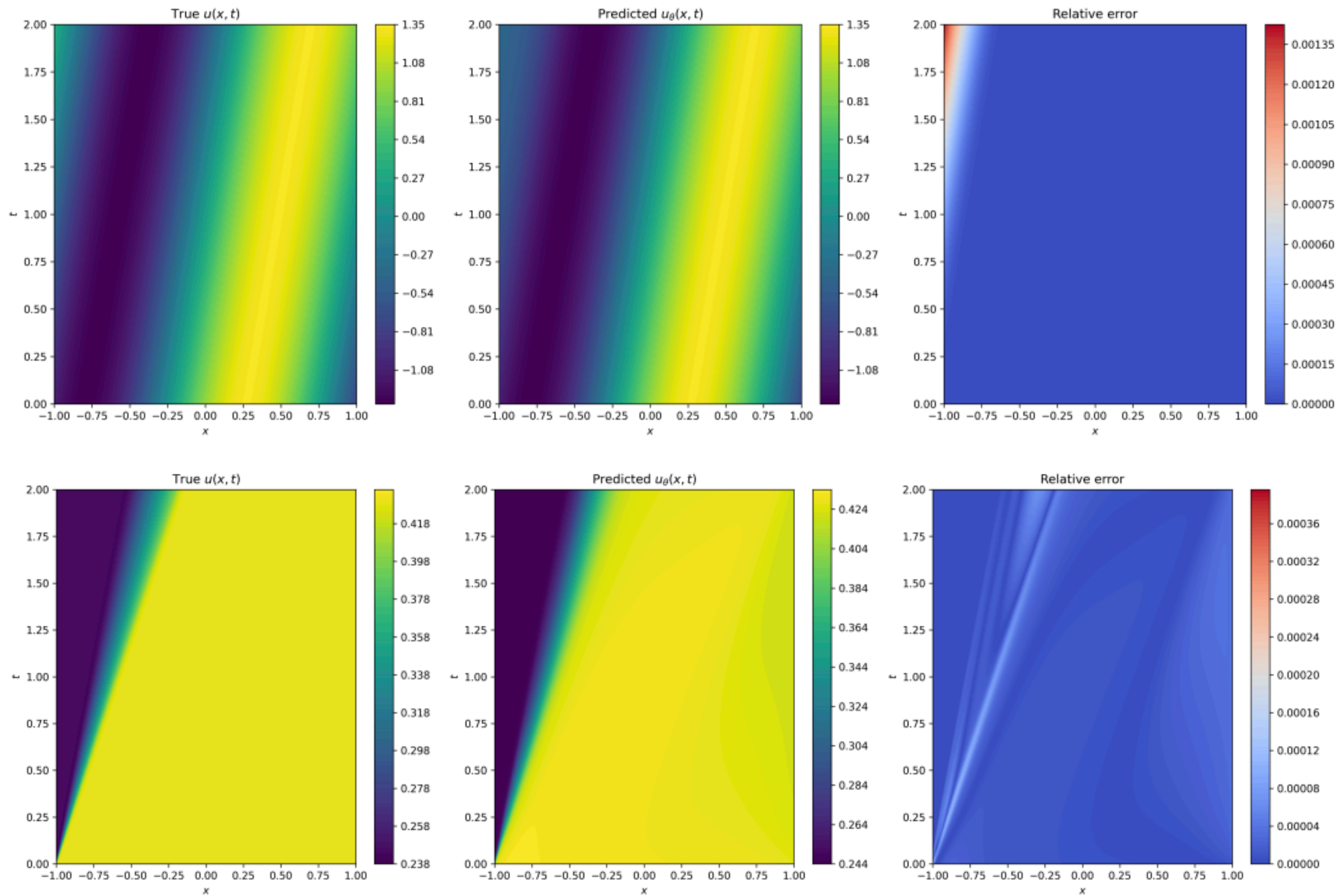
LearnQuad for Family of PDE

$$\mathcal{L}u_i = f_i, \quad \text{in } \Omega$$

$$u_i = g_i, \quad \text{in } \partial\Omega$$



Visual Comparison



Thank you



Exhibit Hall C,D,E

Fri 5 Dec 2025 4:30 p.m. PST — 7:30 p.m. PST