On the Optimality of the Median-of-Means Estimator under Adversarial Contamination

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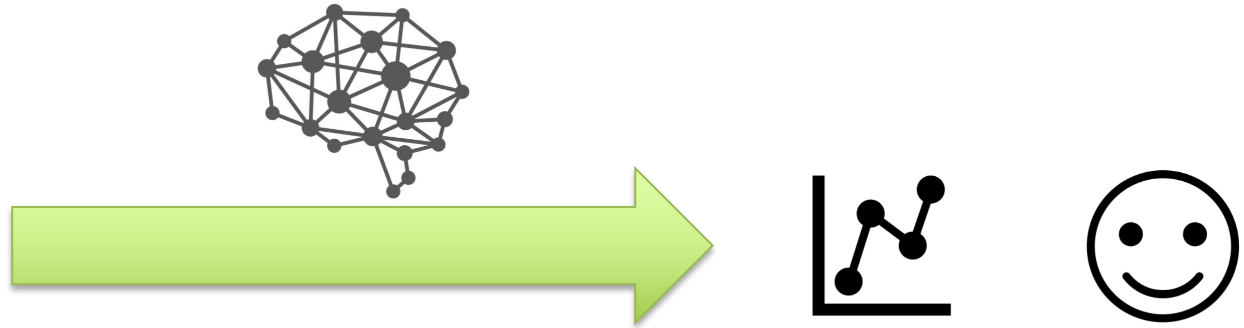
EXCELENCIA
SEVERO
OCHOA

Contaminated data can lead to erroneous decisions

clean data (i.i.d. $\sim \mathrm{p}$)

11	9	12	10	7



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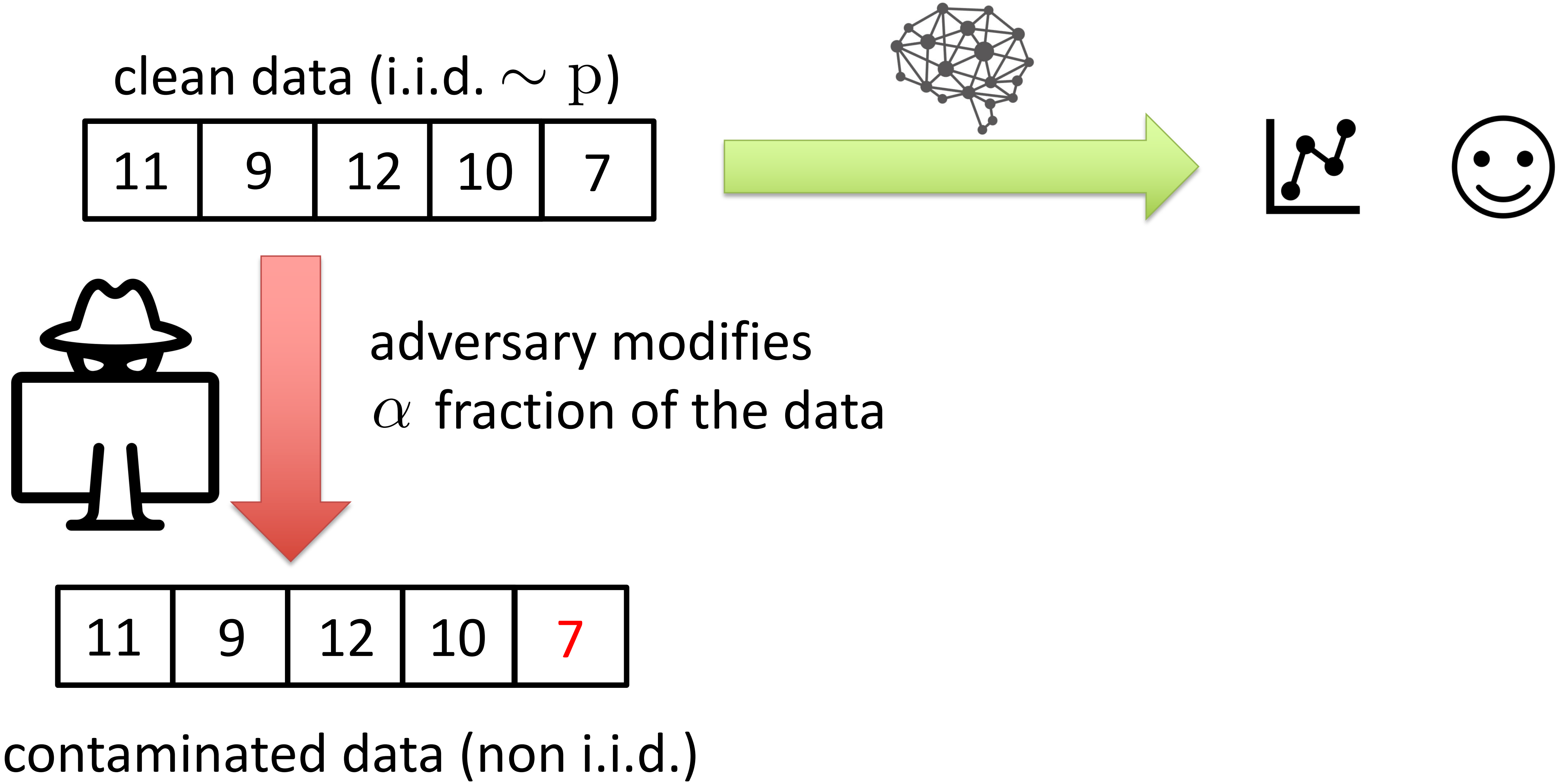

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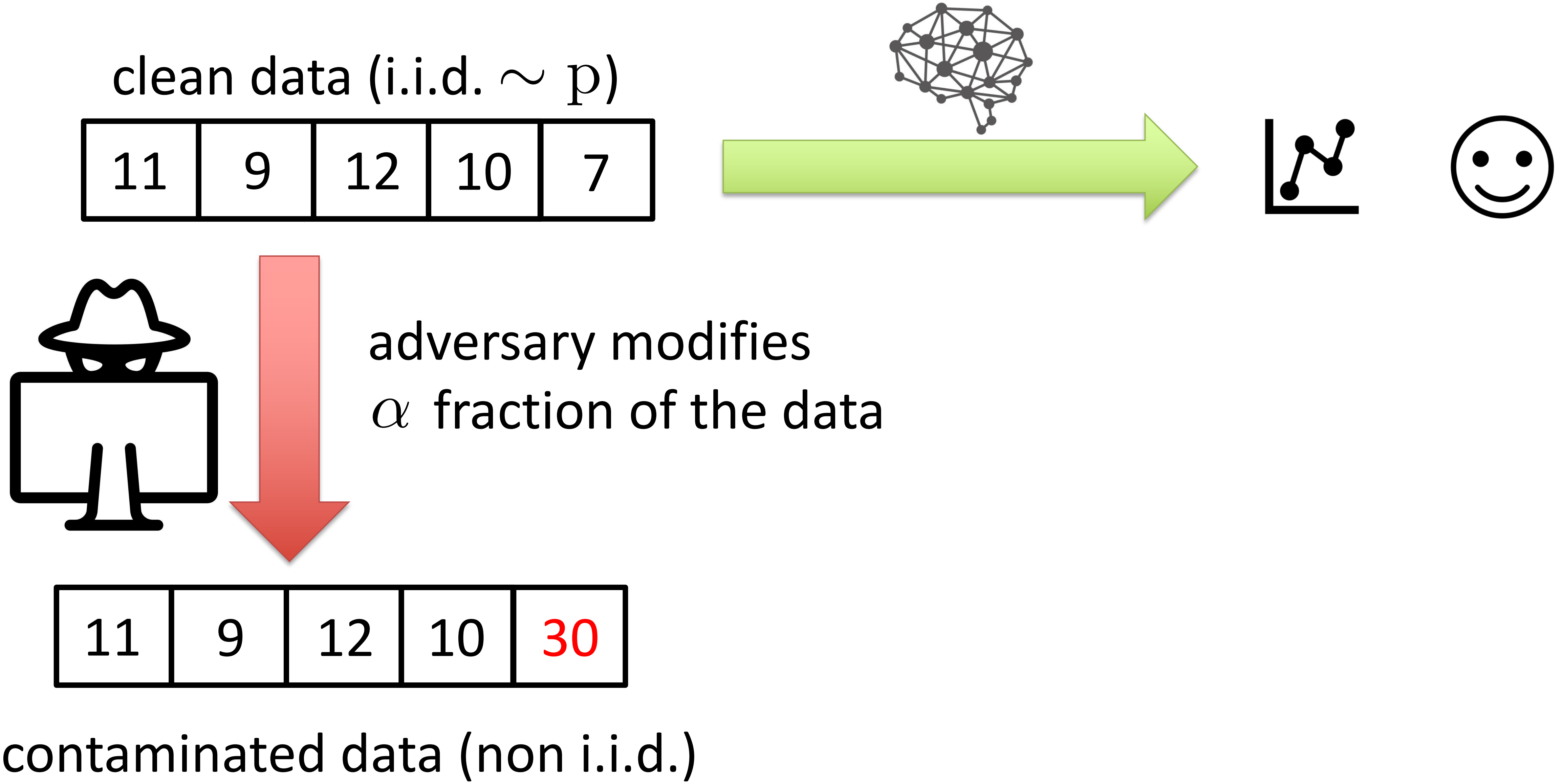

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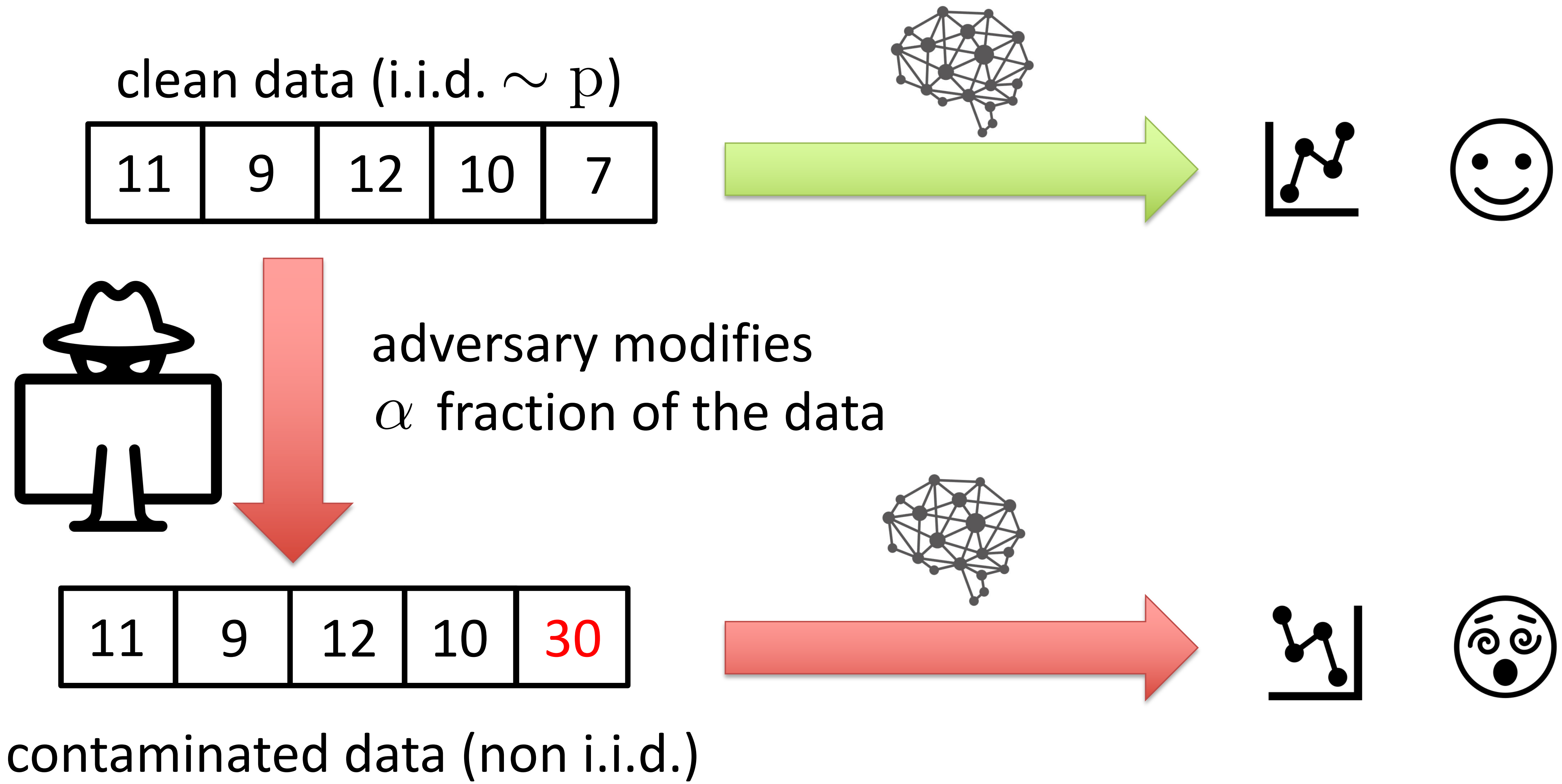

Mean estimation under adversarial contamination

Mean of the clean distribution

Class of distributions

$$|\widehat{\mu} - \mu_{\mathrm{p}}| \leq \sigma_{\mathrm{p}} \varepsilon(\alpha) \quad \text{with high probability} \quad \forall \mathrm{p} \in \mathcal{P}$$

Estimator of the mean

Asymptotic bias (the cost of contamination)

$$\varepsilon(\alpha) \not\to 0 \quad \text{when} \quad n \to +\infty$$

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Previous work on the Median-of-Means (MoM)

- Lacks analysis of the asymptotic bias
- Focuses only on the class of distributions with finite variance
- Considers a weaker contamination model

Contributions

	MoM [this work]	Trimmed mean	M-estimator
Heavy-tailed (finite variance)	$\sqrt{\alpha}$	$\sqrt{\alpha}$	$\sqrt{\alpha}$
Heavy-tailed (infinite variance)	$\alpha^{\frac{r}{1+r}}$		$\alpha^{\frac{r}{1+r}}$
Light-tailed (sub-exponential)	$\alpha^{2/3}$	$\alpha\sqrt{\log(1/\alpha)}$	

MoM is optimal for heavy-tailed distributions

Theorem. Let $\widehat{\mu}_{\mathrm{MoM}}$ be MoM with a number of blocks $k = \mathcal{O}(\alpha n)$

Minimax optimal!

- If $\mathrm{p} \in \mathcal{P}_2$, i.e., finite variance

  $$|\widehat{\mu}_{\mathrm{MoM}} - \mu_{\mathrm{p}}| \leq \mathcal{O}(\sqrt{\alpha}) \quad \text{with high probability}$$

- If $\mathrm{p} \in \mathcal{P}_{1+r}$, i.e., finite $(1+r)$-th moment

  $$|\widehat{\mu}_{\mathrm{MoM}} - \mu_{\mathrm{p}}| \leq \mathcal{O}\left(\alpha^{\frac{r}{1+r}}\right) \quad \text{with high probability}$$

MoM is sub-optimal for light-tailed distributions

Theorem. Let $\widehat{\mu}_{\mathrm{MoM}}$ be MoM with a num. of blocks $k = \mathcal{O}(\alpha^{2/3}n)$

Then, for any light-tailed distribution p

$$|\widehat{\mu}_{\mathrm{MoM}} - \mu_{\mathrm{p}}| \leq \mathcal{O}\left(\alpha^{2/3}\right)$$ with high probability.

Sub-optimal!

Theorem. There is a light-tailed distribution & attack s.t. for any k

$$|\widehat{\mu}_{\mathrm{MoM}} - \mu_{\mathrm{p}}| \geq \mathcal{O}\left(\alpha^{2/3}\right)$$ with high probability