Provable Speedups From Dynamic Population Sizes in Evolutionary Algorithms for Multiobjective Optimization

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ThRaSH Seminar Autumn 2026



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Overall goal: Find improved variants of NSGA-II to improve performance

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Paper accepted at NEURIPS 2026!

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NSGA-II - population size $\mu = |P_t|$

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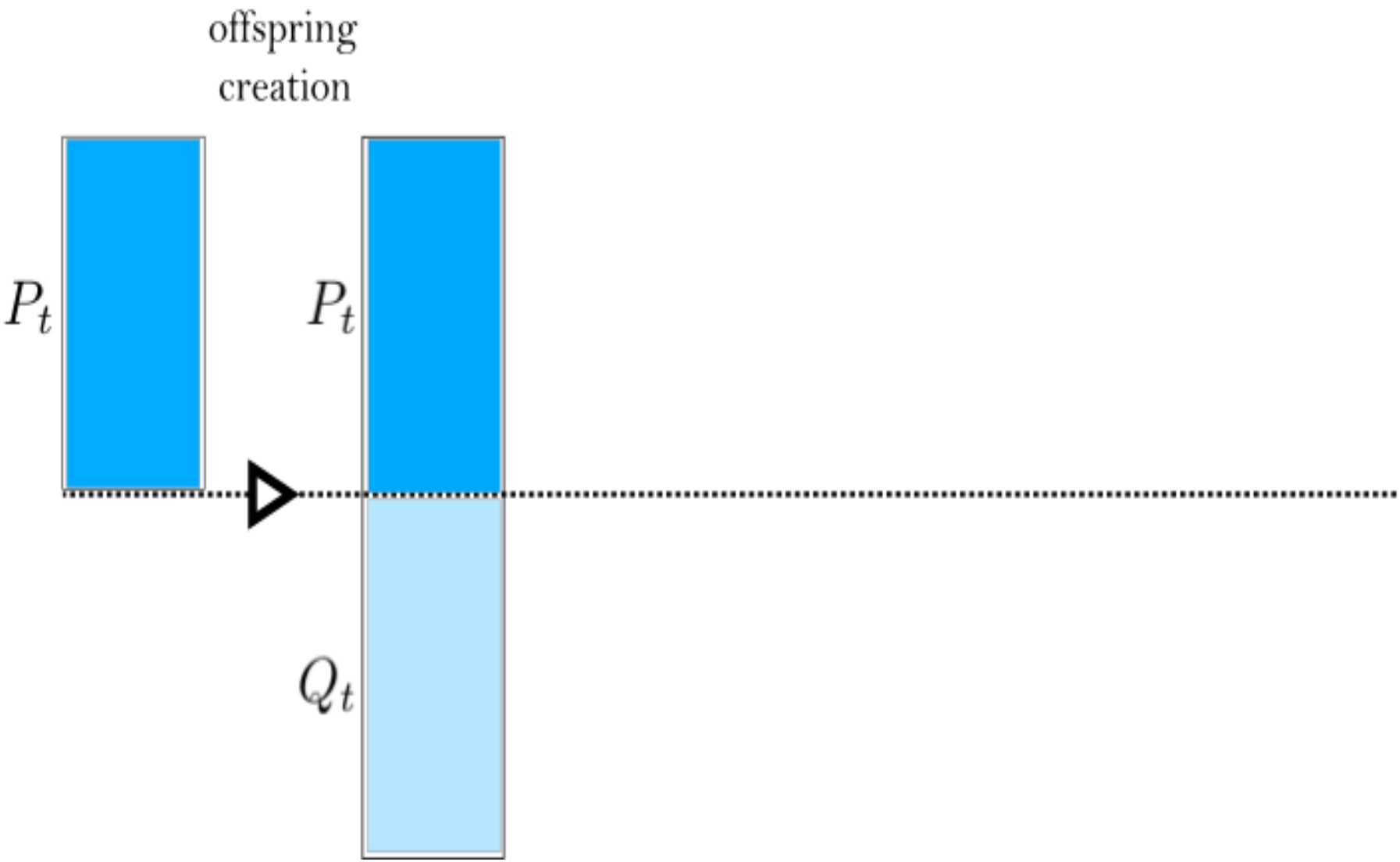
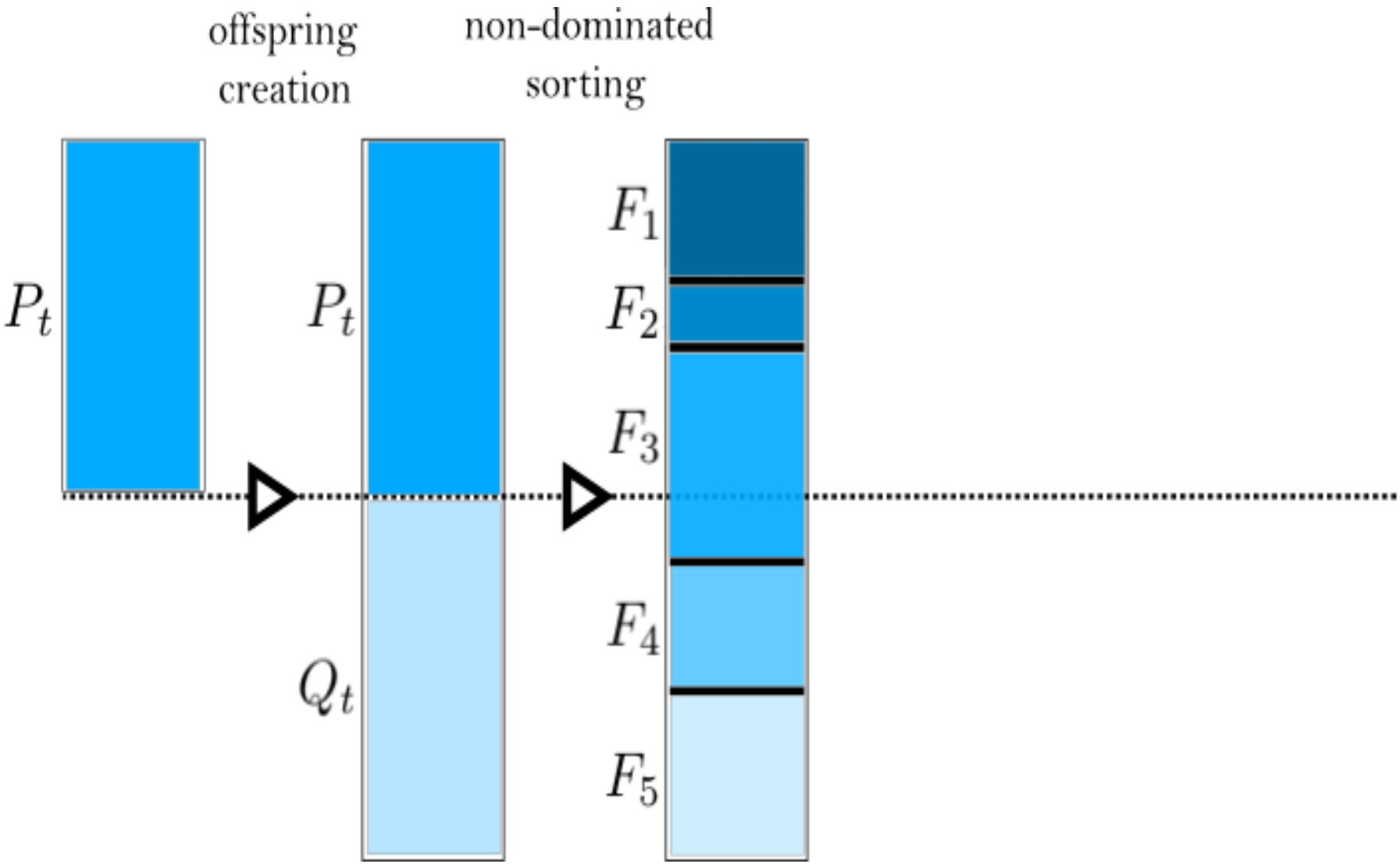


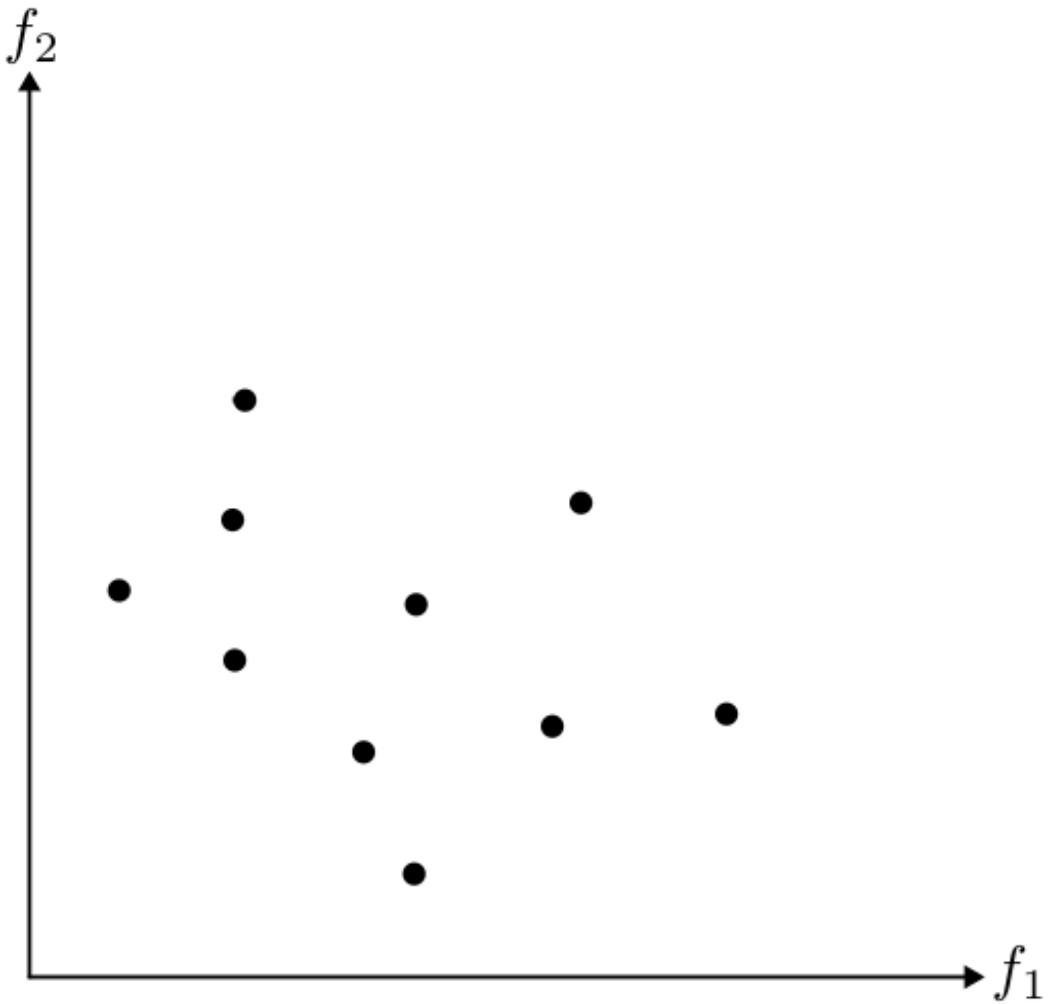
NSGA-II - creating $\mu = |Q_t|$ offspring

Many-Objective Optimisation



NSGA-II algorithm - non-dominated sorting

Non-Dominated Sorting



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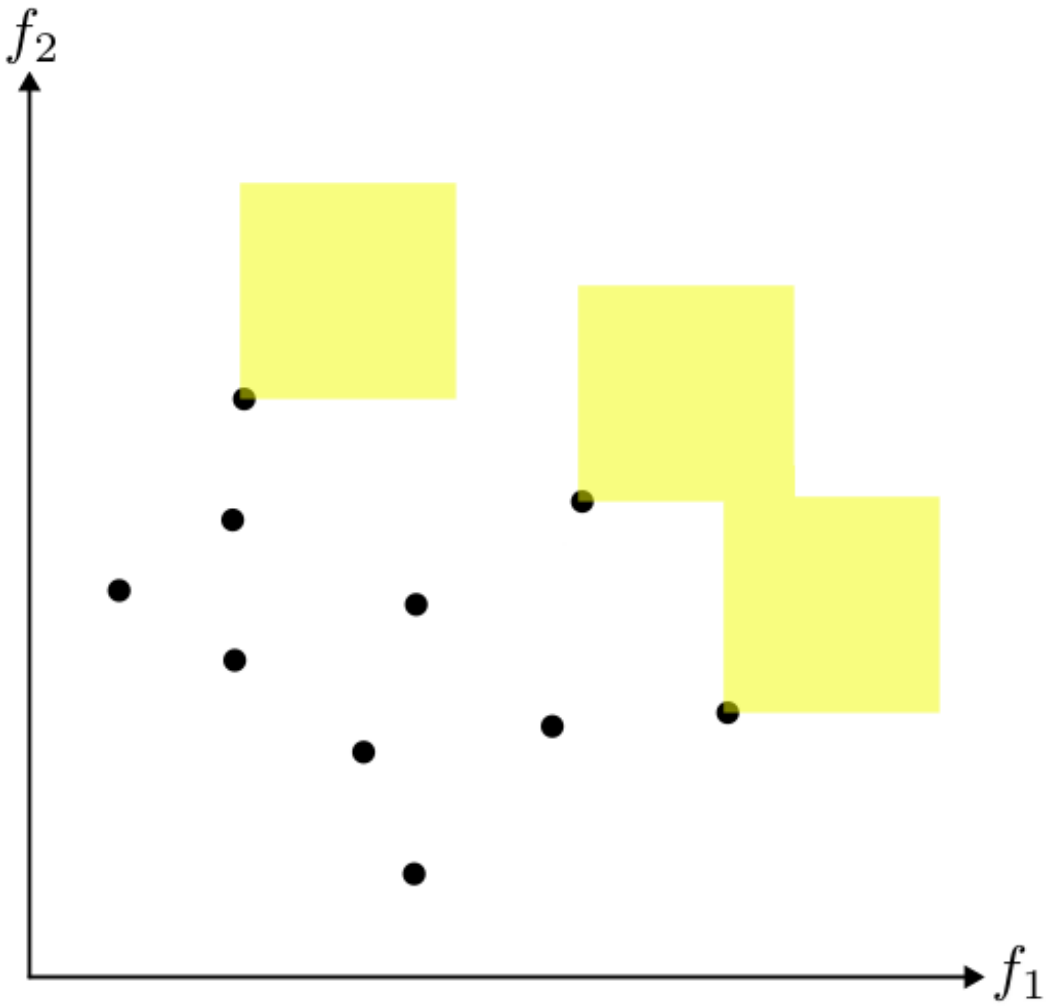

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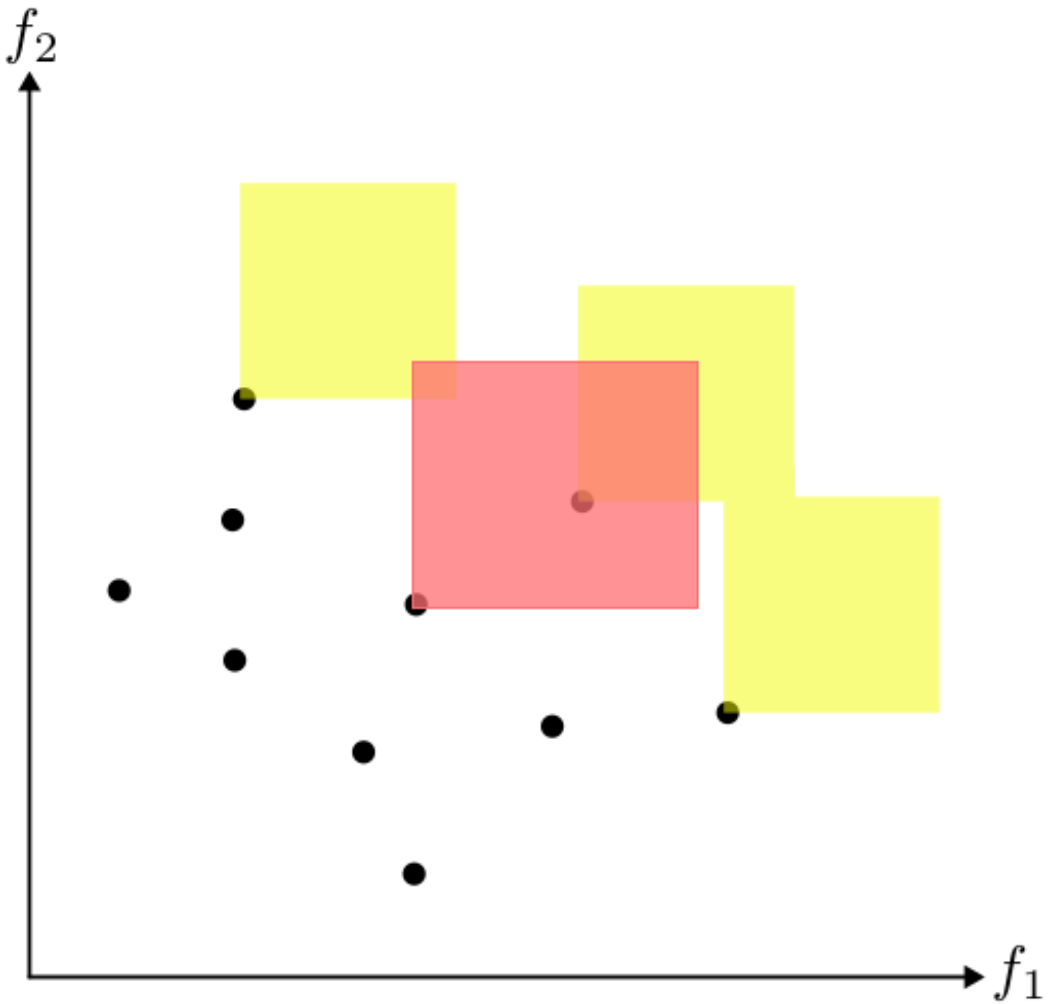

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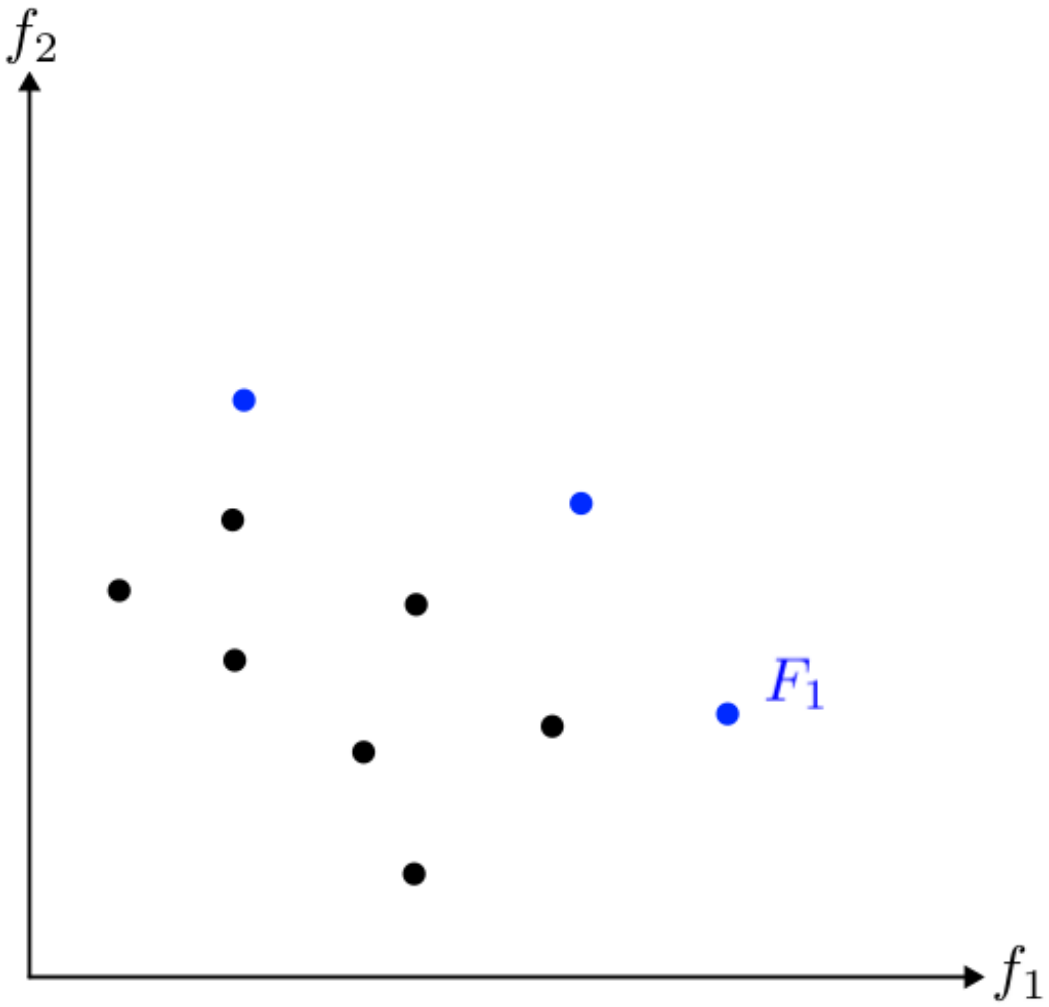

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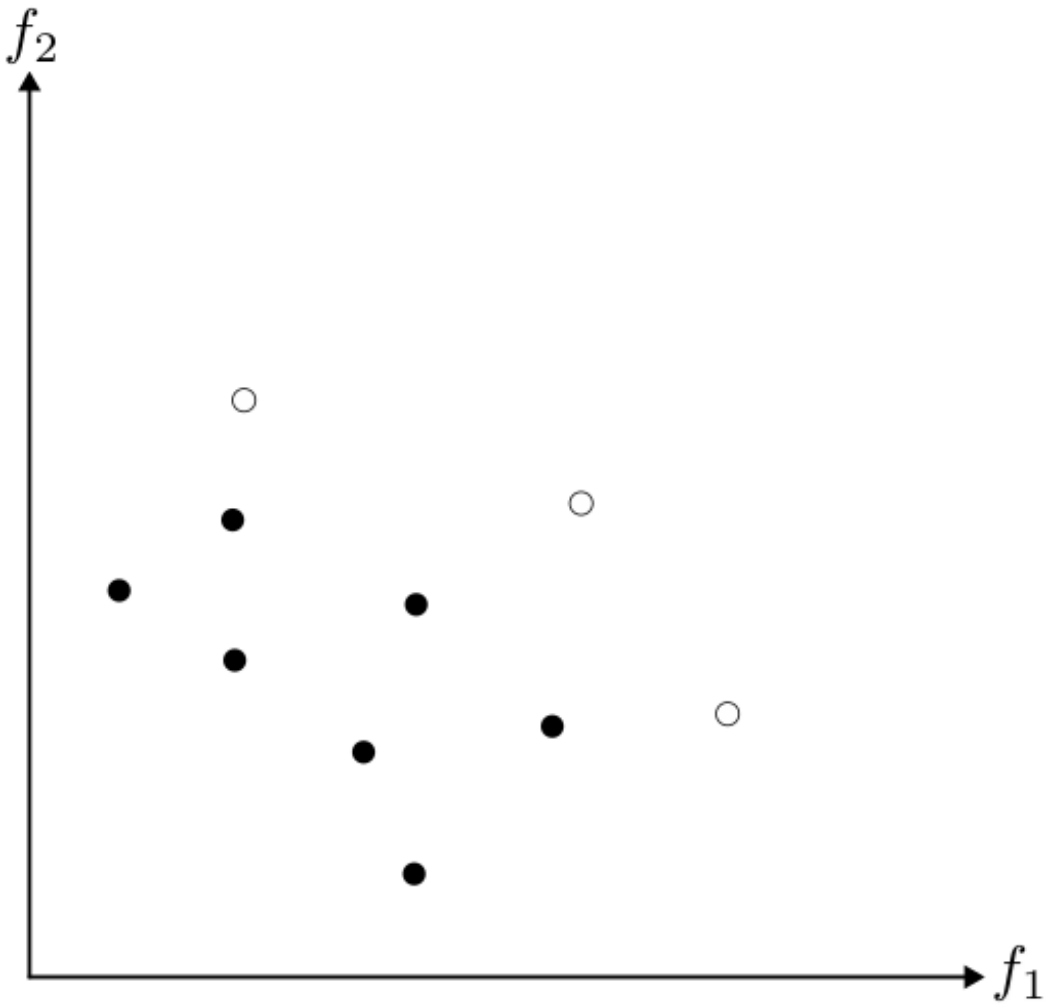

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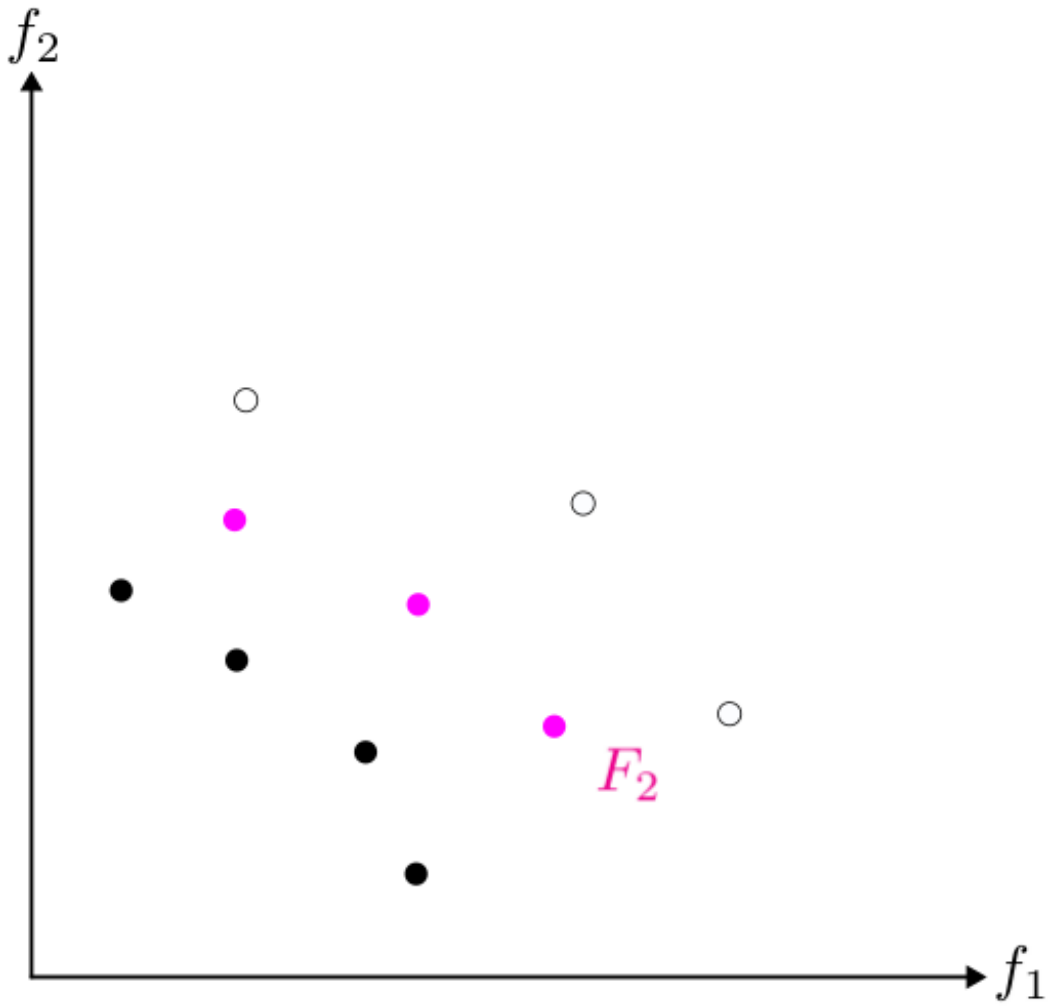

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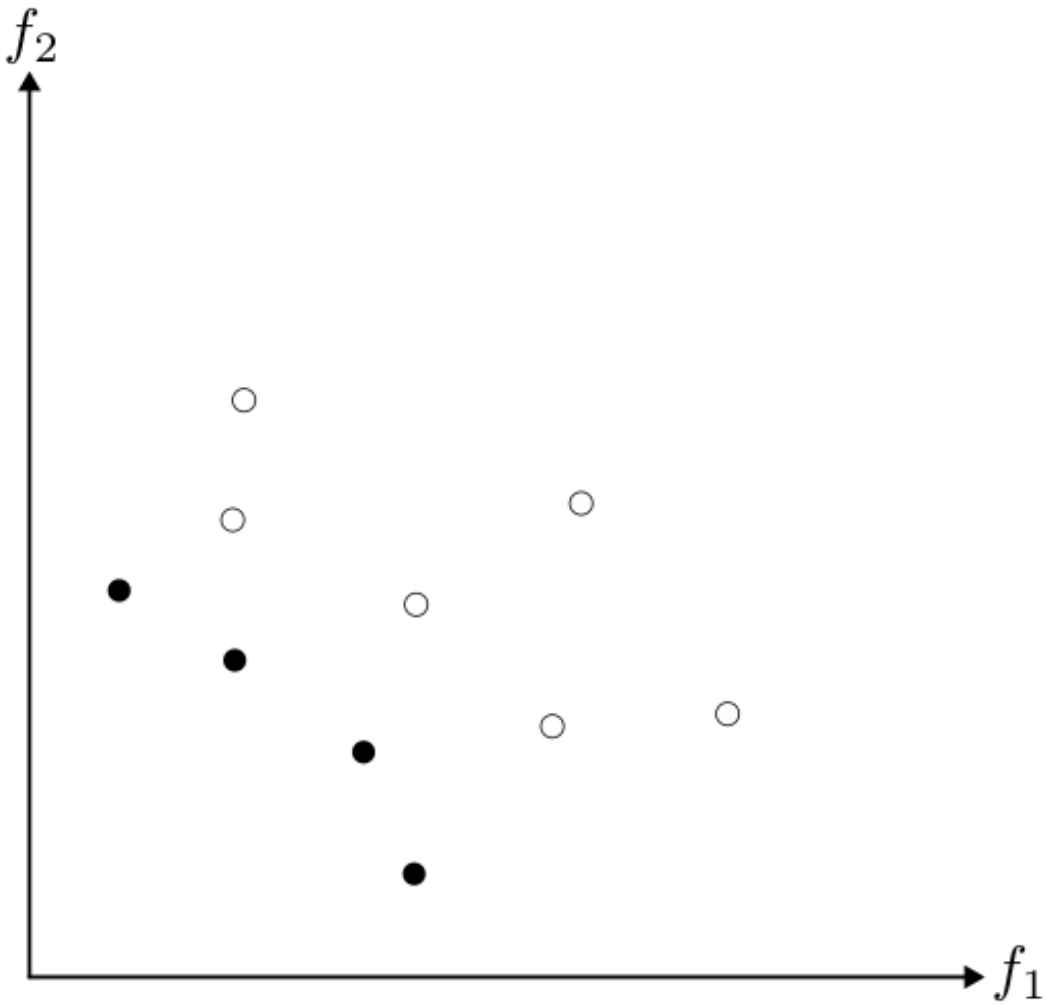

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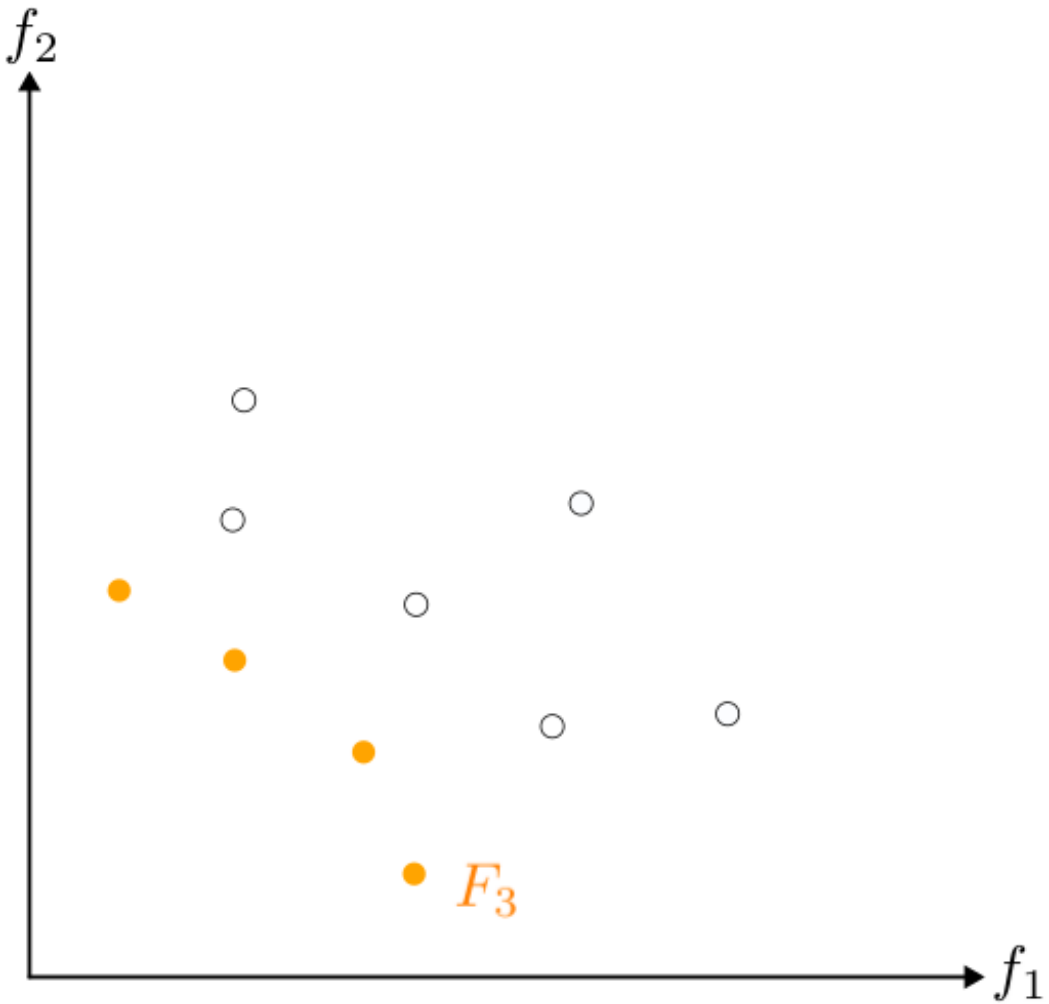

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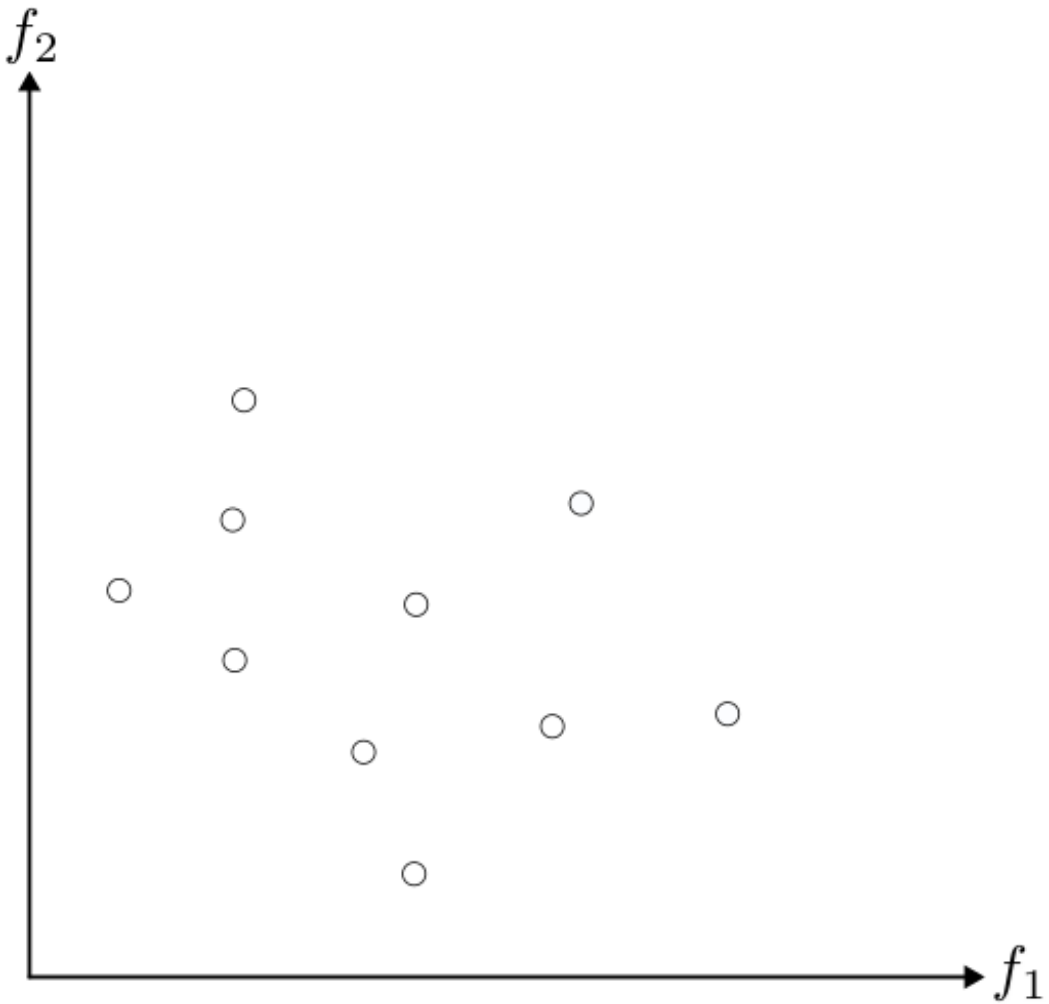

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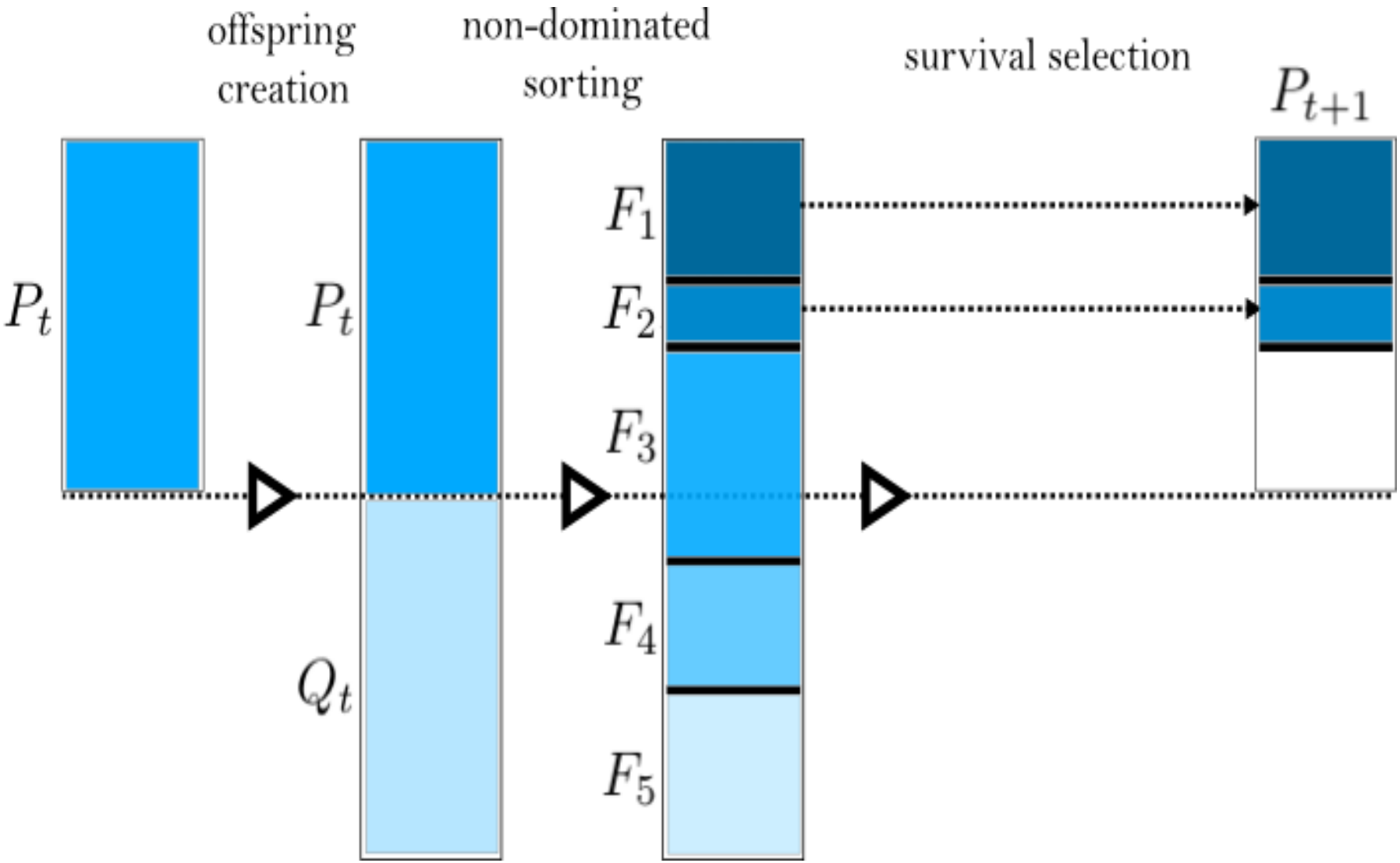
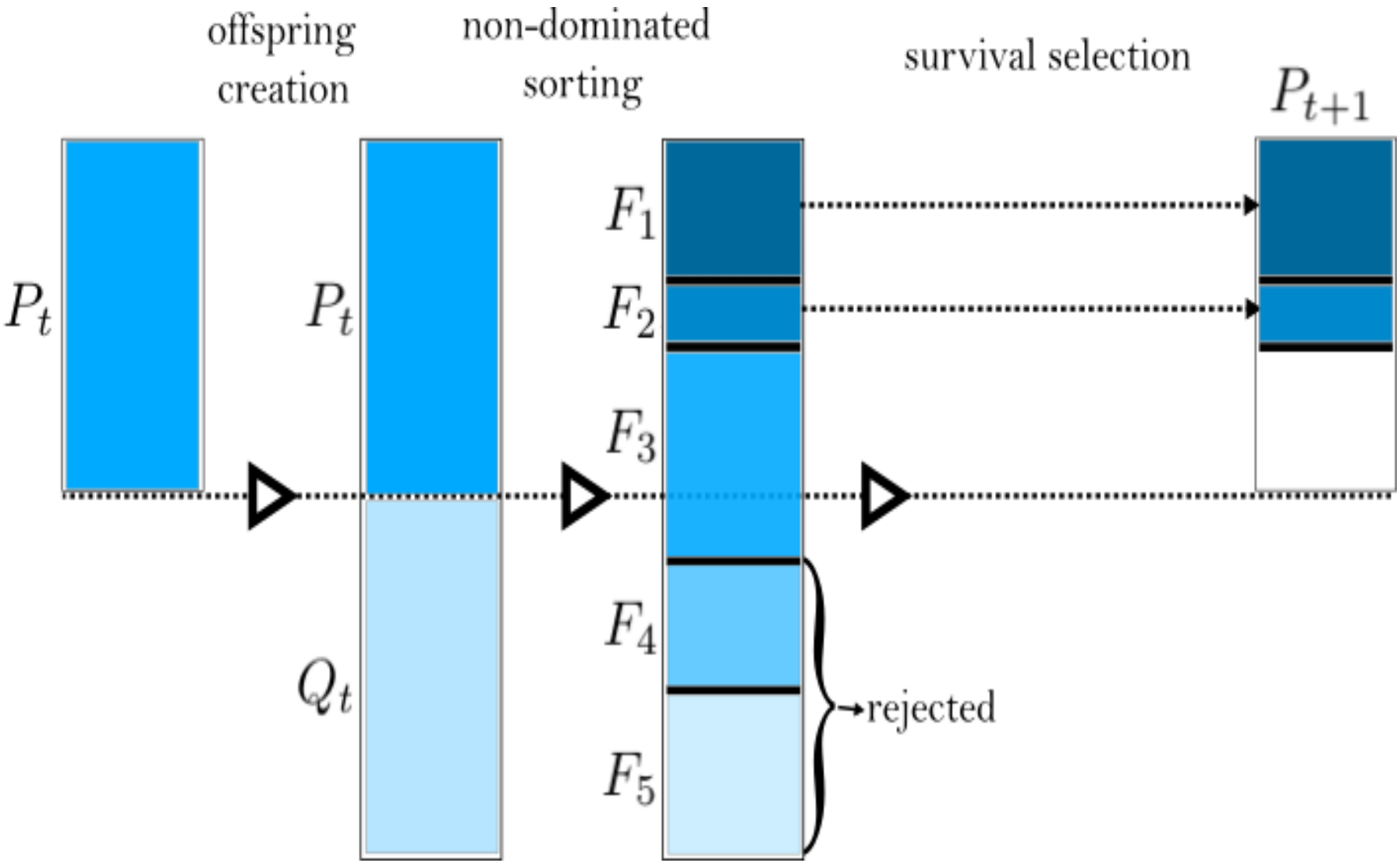


NSGA-II algorithm - survival selection



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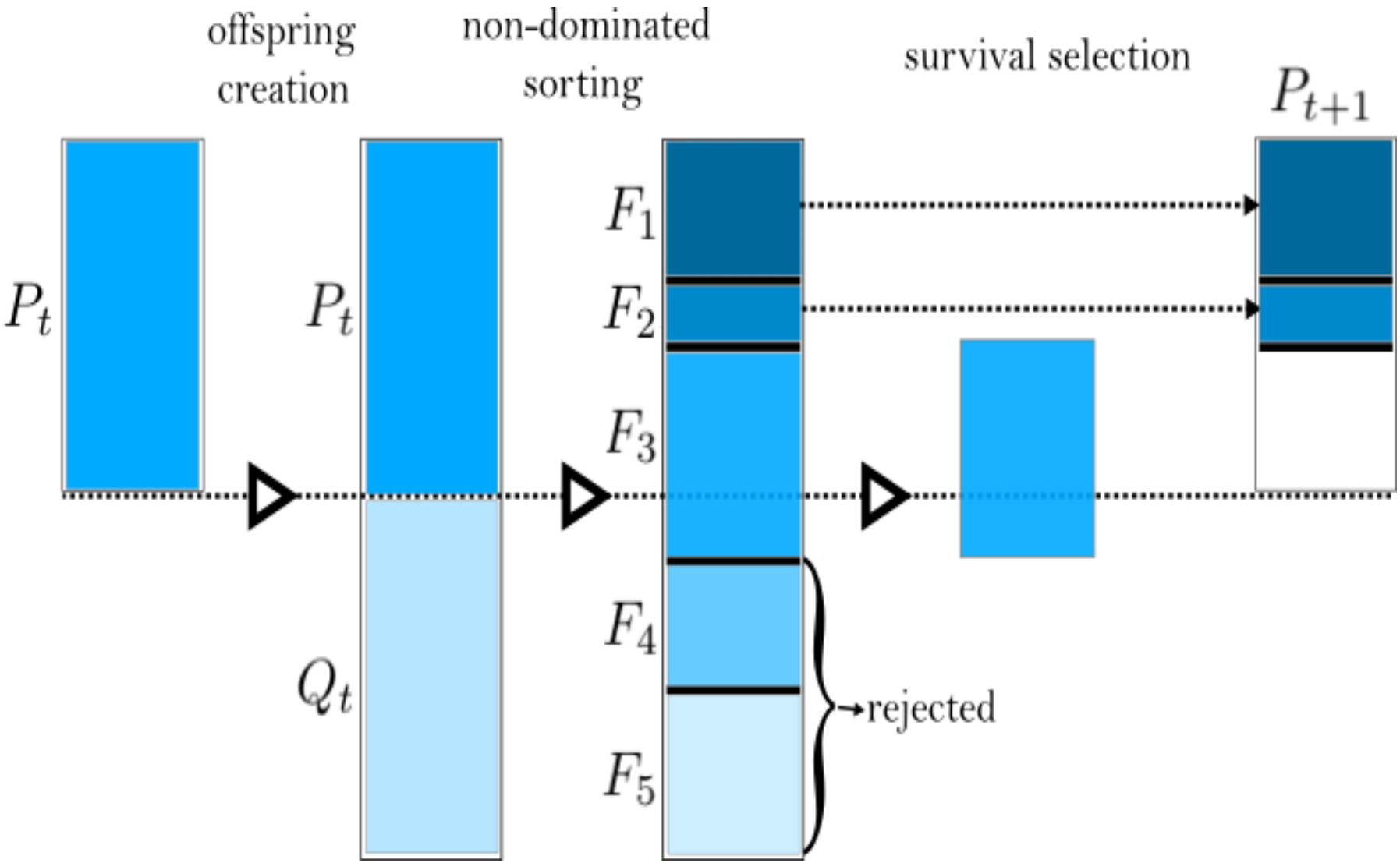

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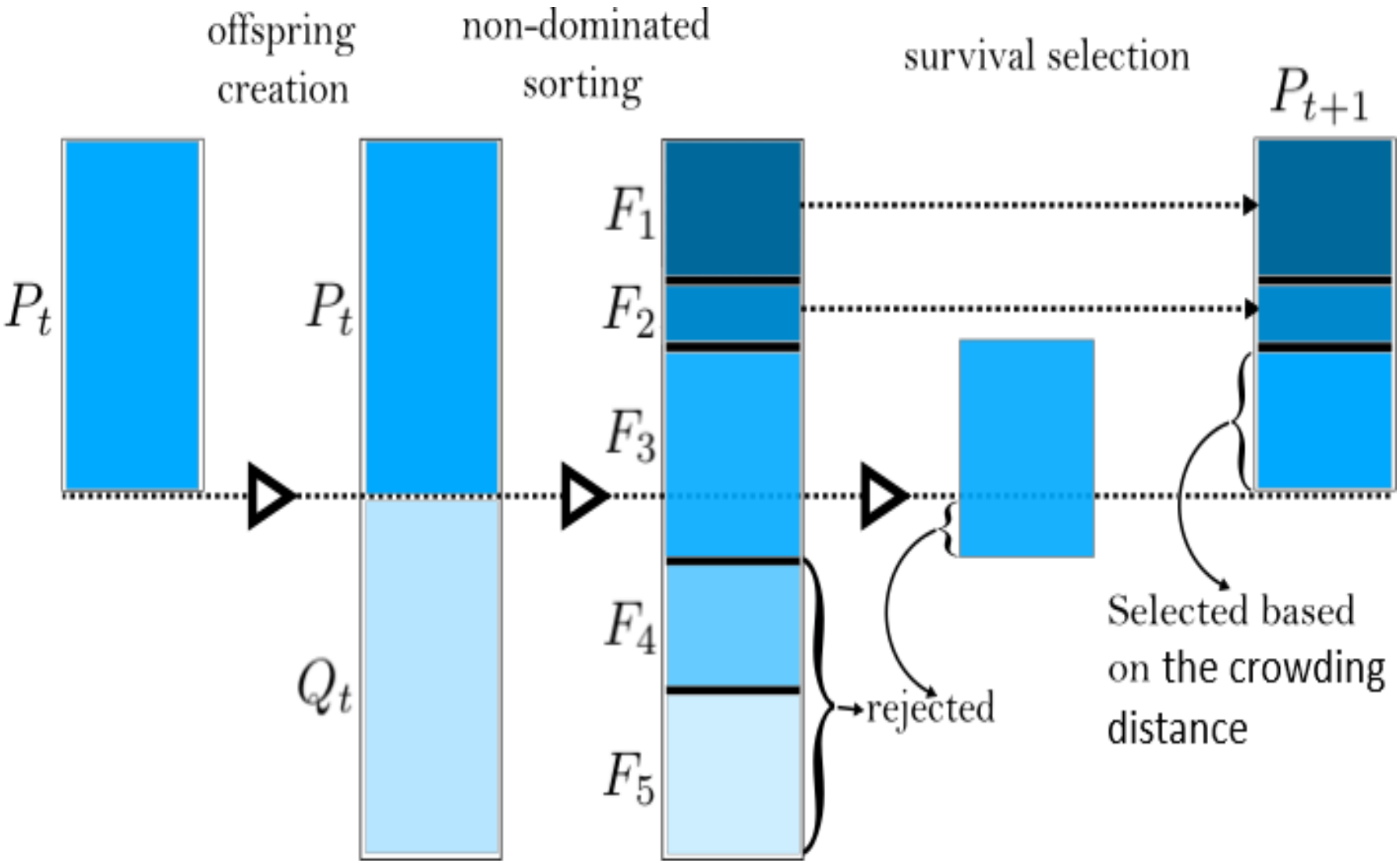

NSGA-II algorithm - Survival Selection

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Estimate distance between nearby solutions, favor less crowded areas

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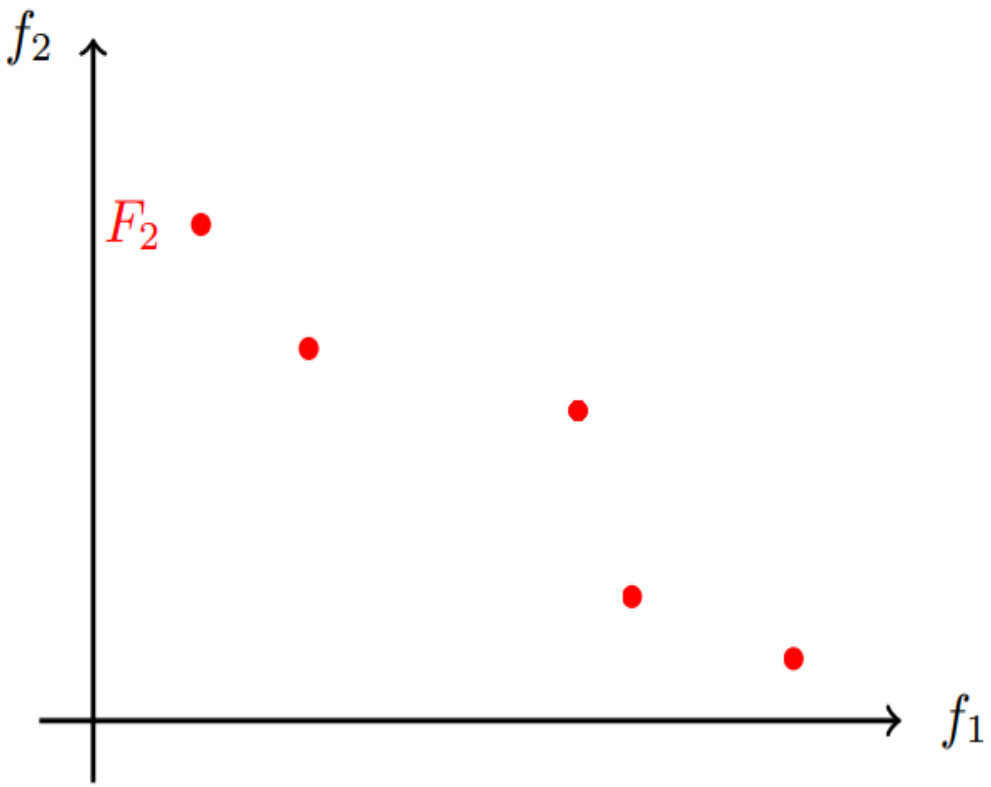

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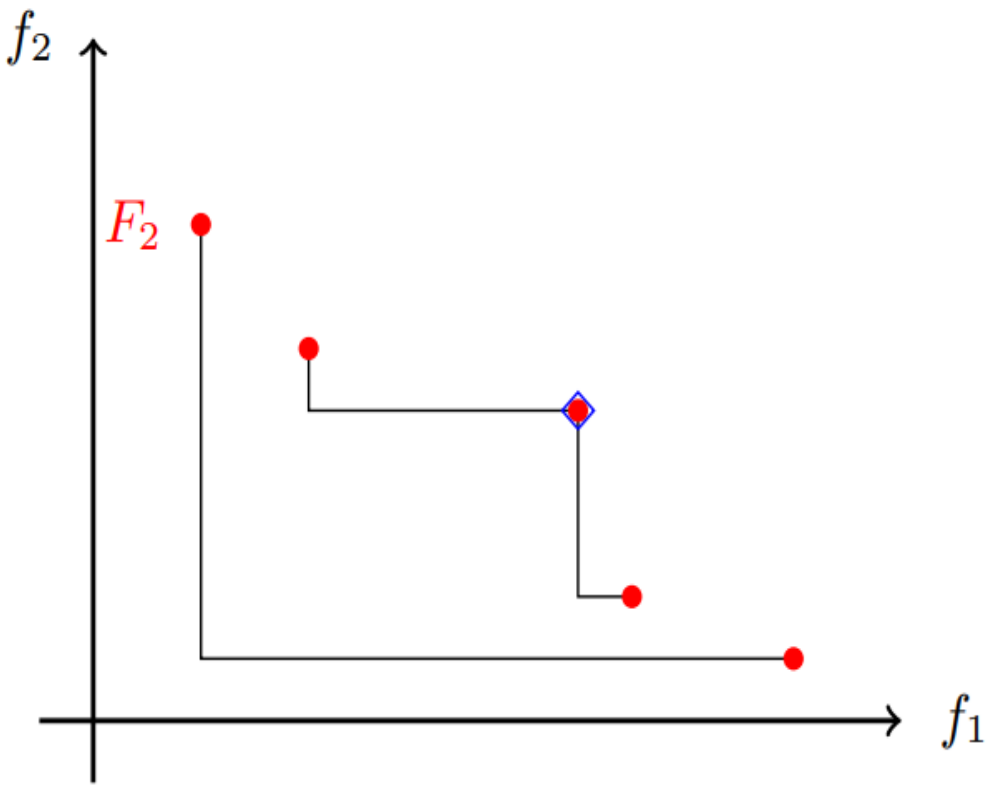

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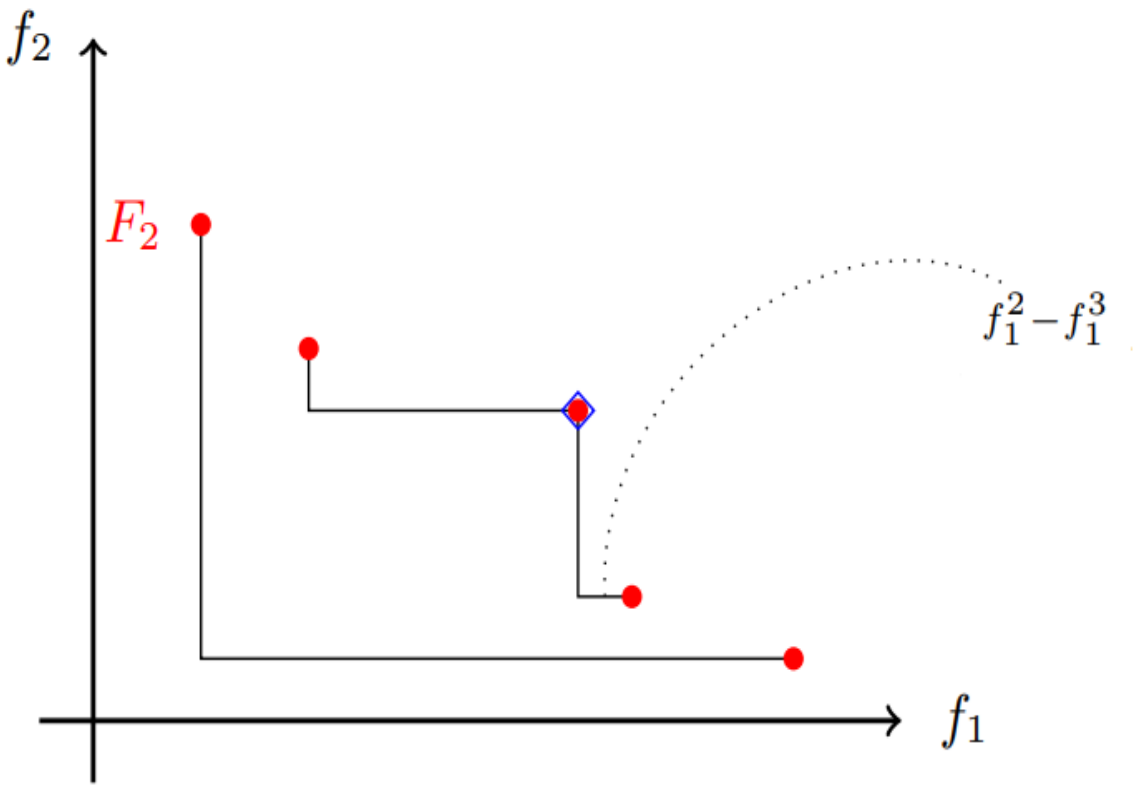

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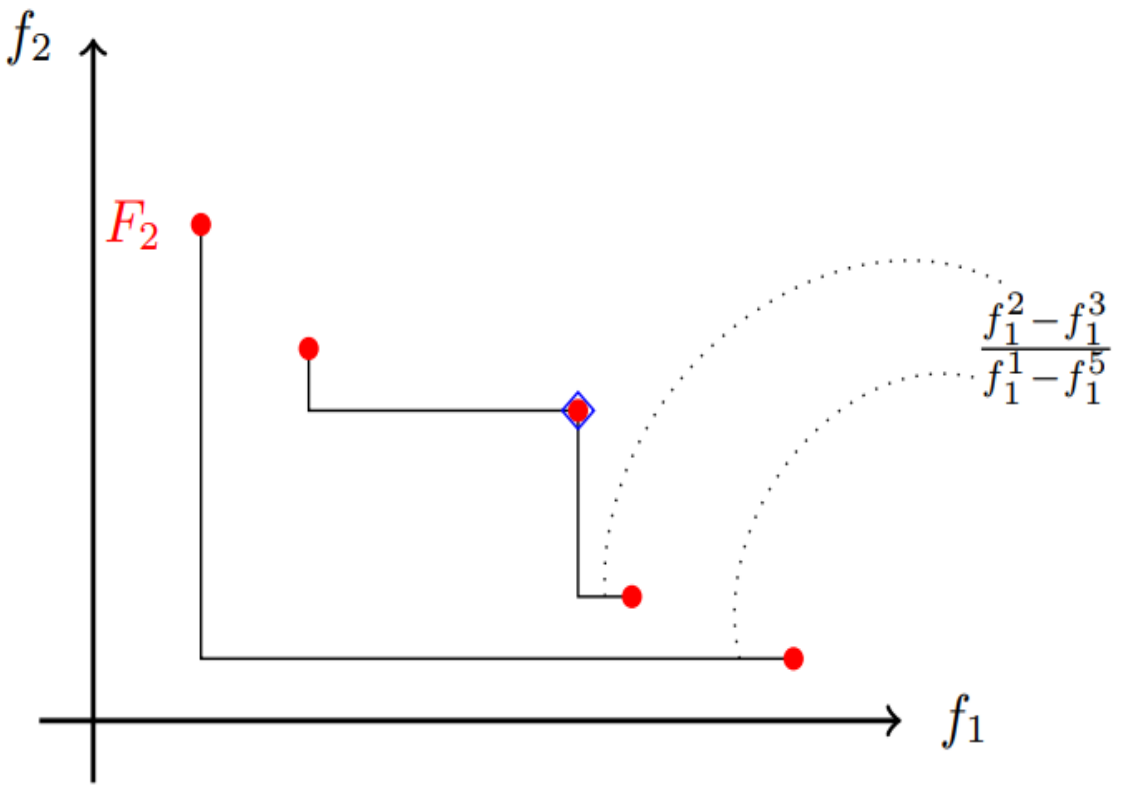

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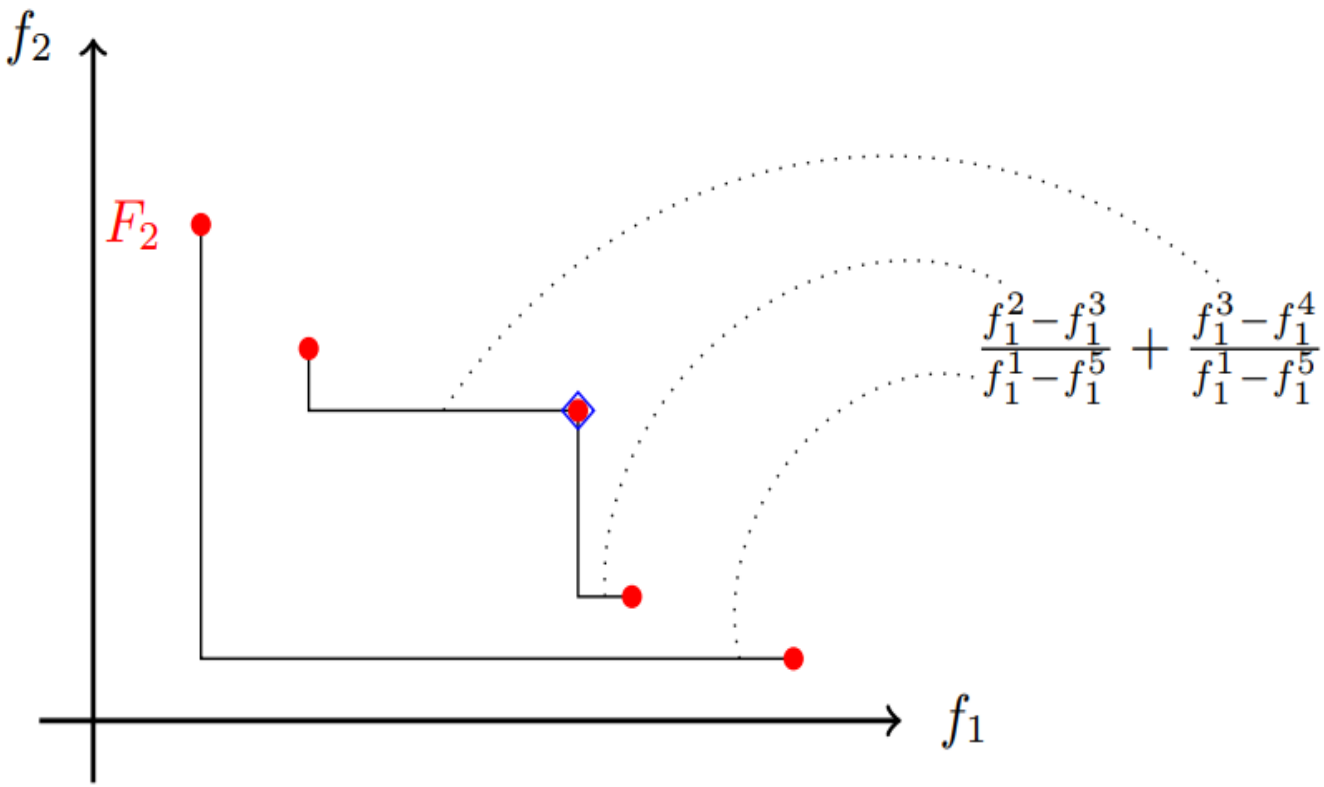

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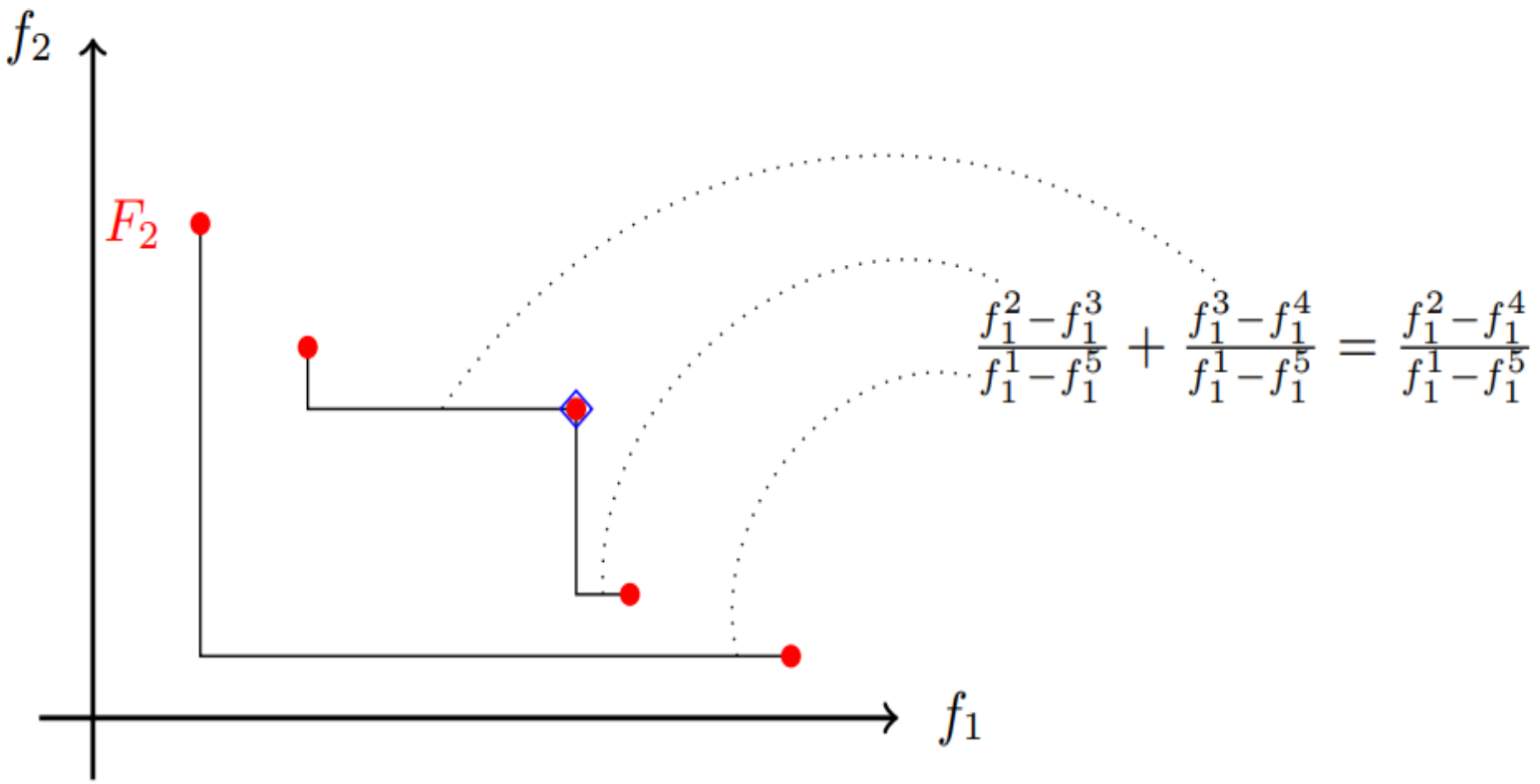

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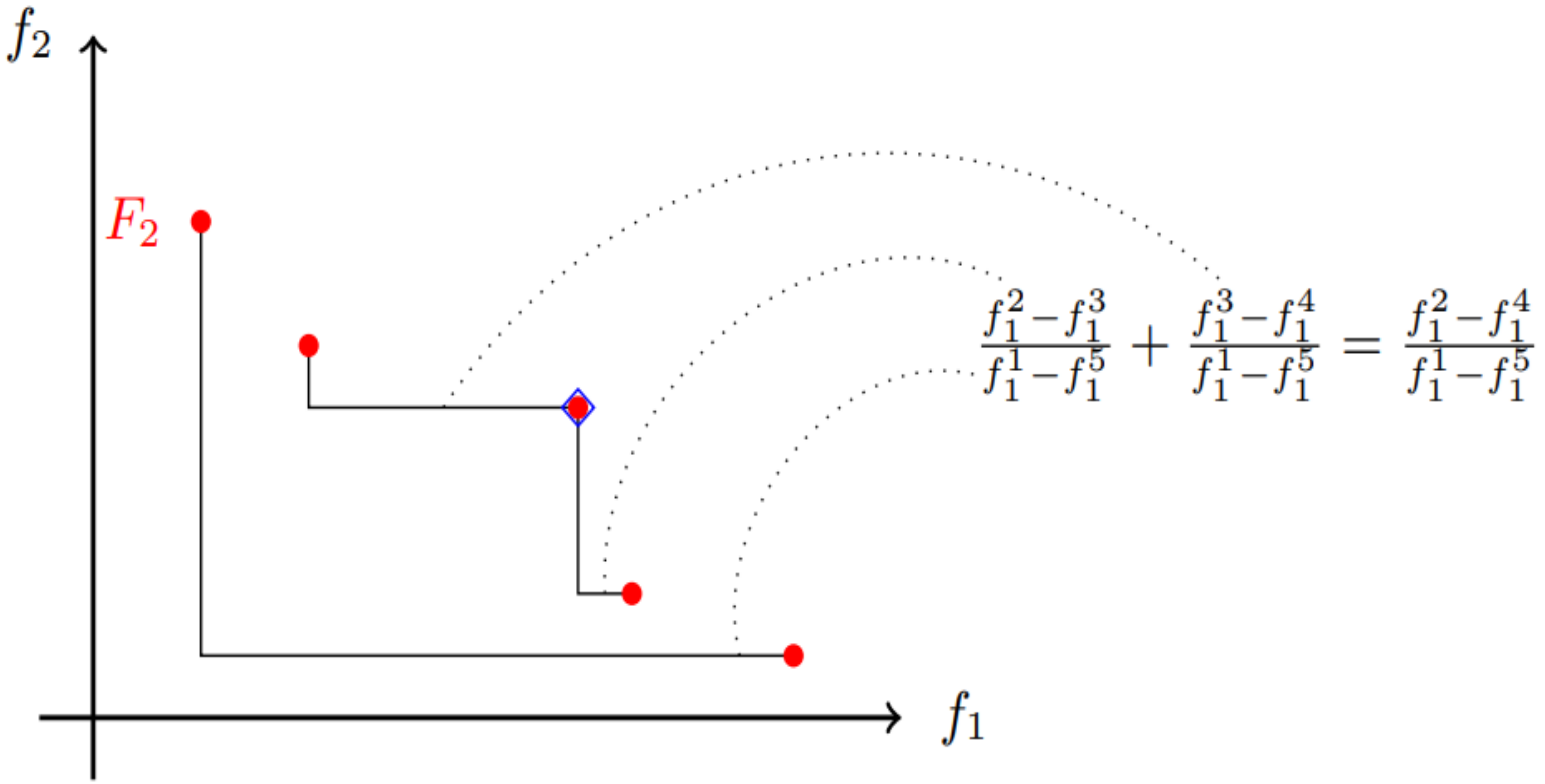
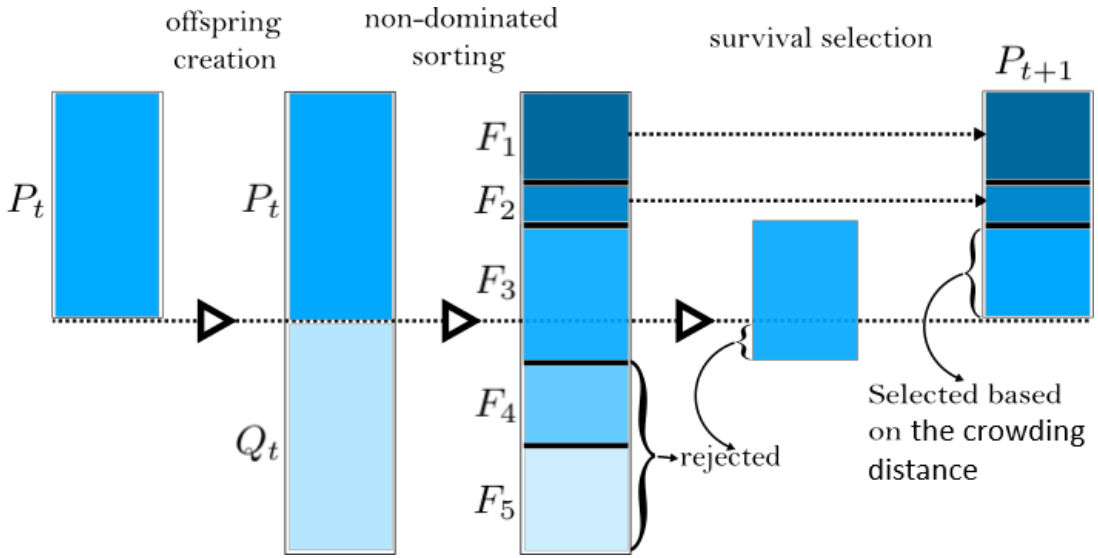
  - (iii) For two extreme solutions $x_{I[1]}$ and $x_{I[\mu]}$, add crowding distance ∞



Many-Objective Optimisation



NSGA-II algorithm

Note: NSGA-II has a fixed population size!

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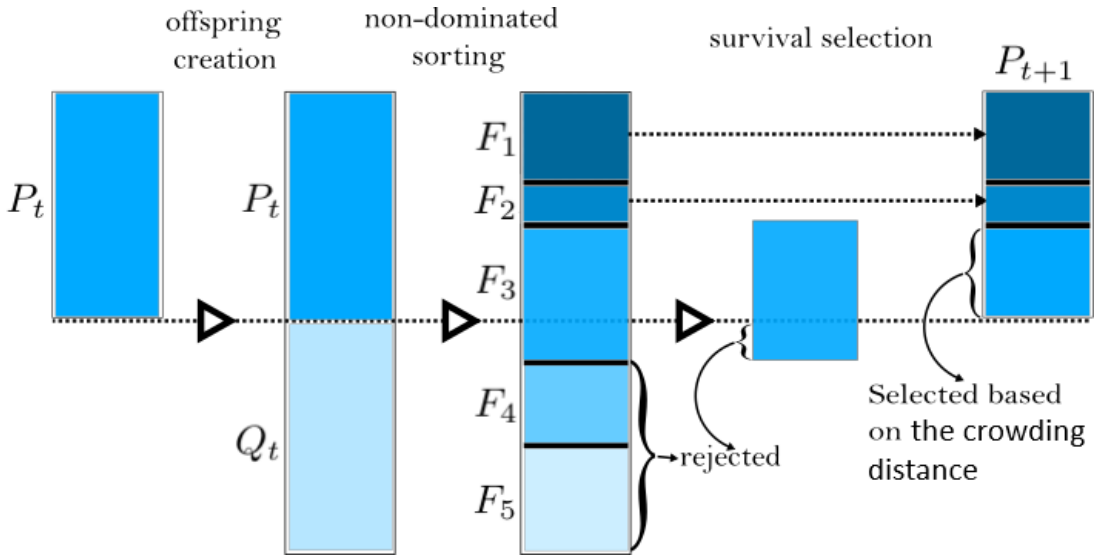

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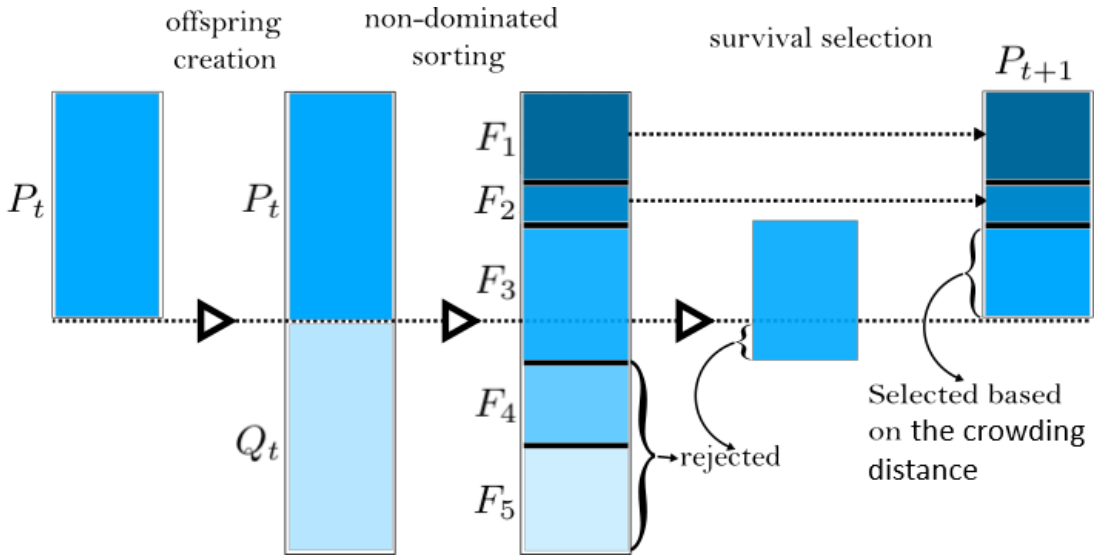

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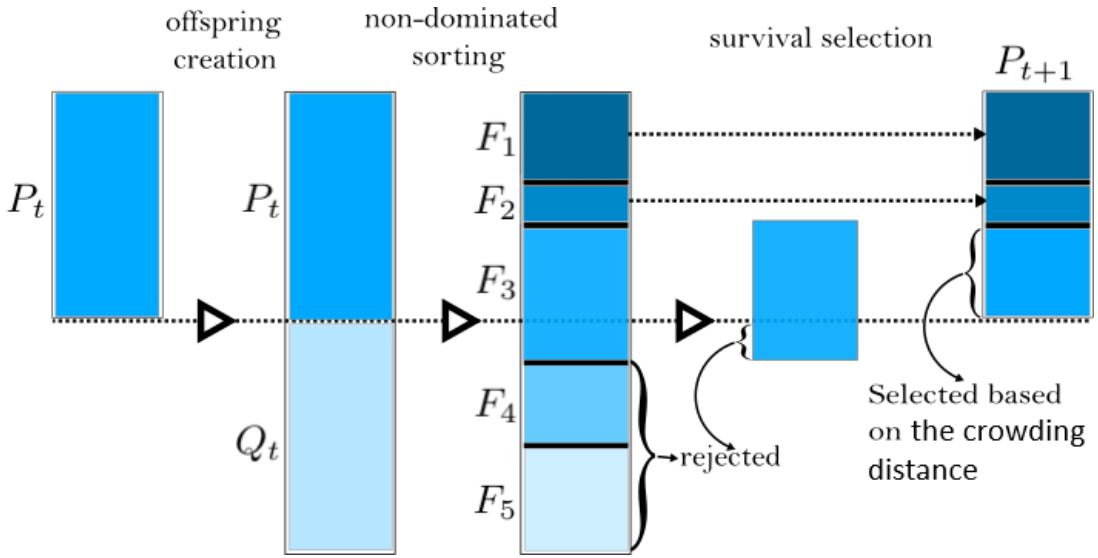

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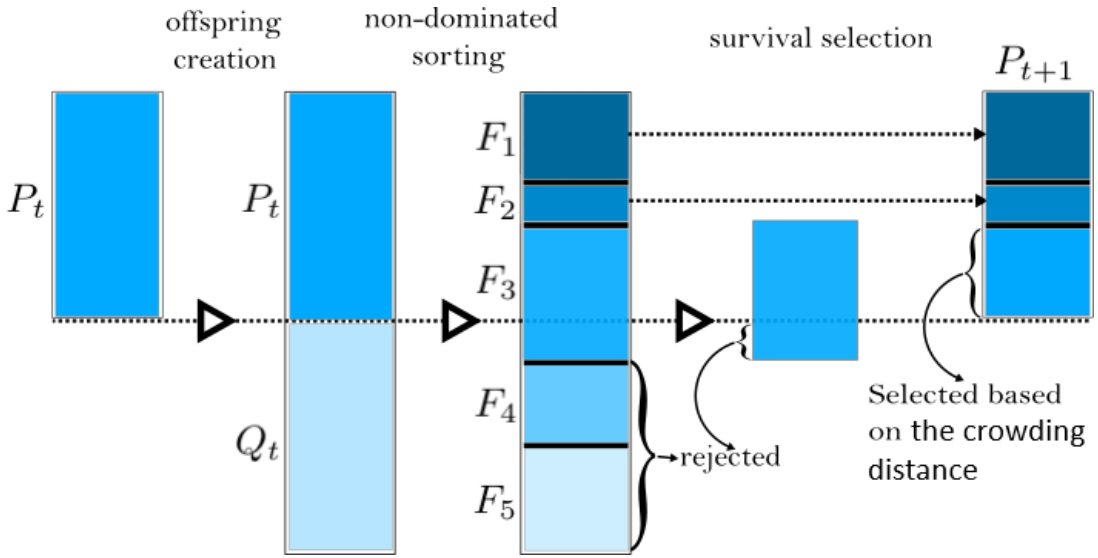

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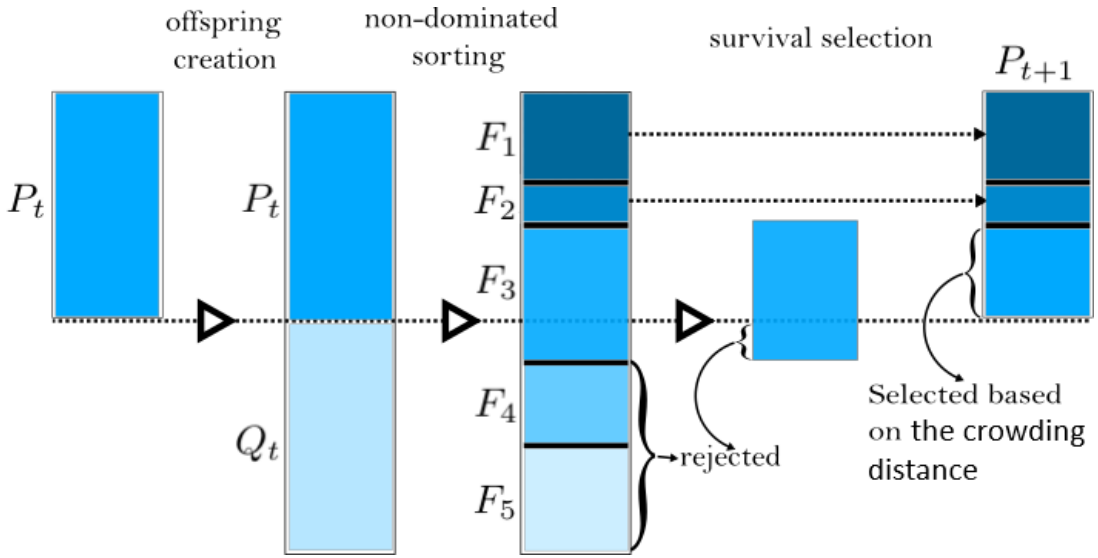

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Requirements:

- Let n be a natural number divisible by 4

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Parameter ℓ: Parameterizes size of Pareto front

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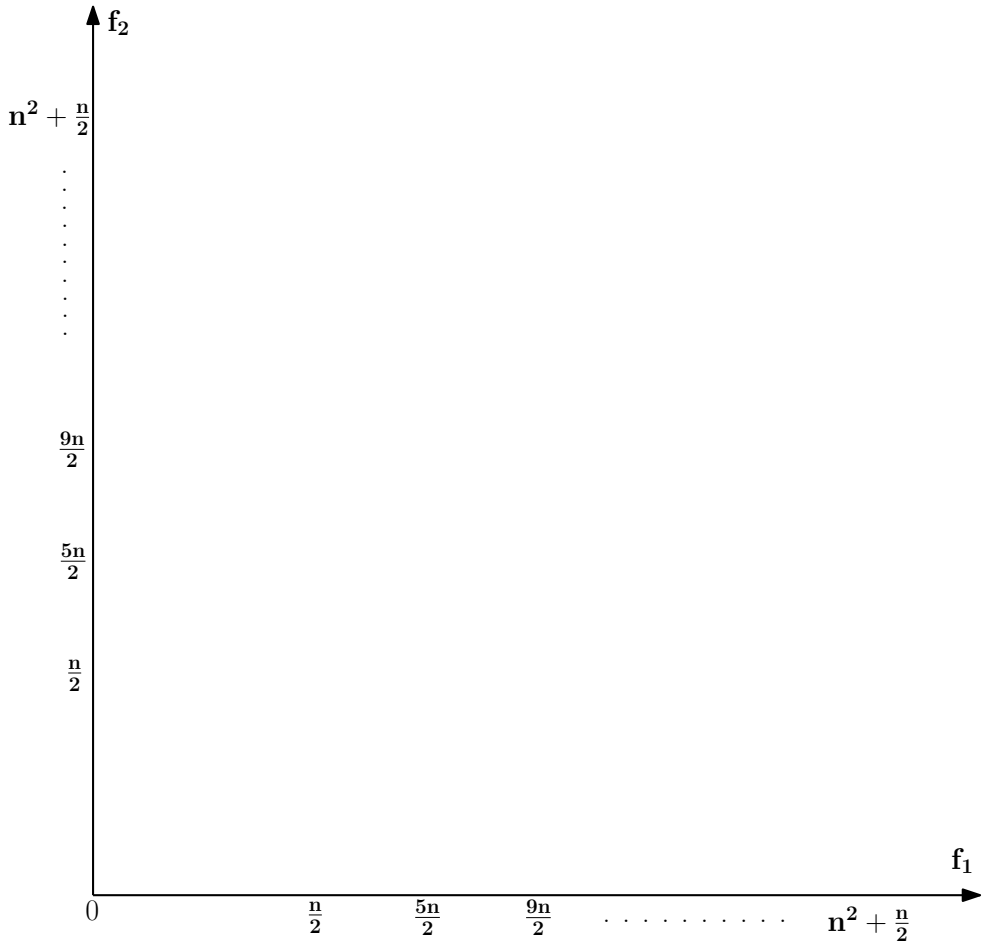

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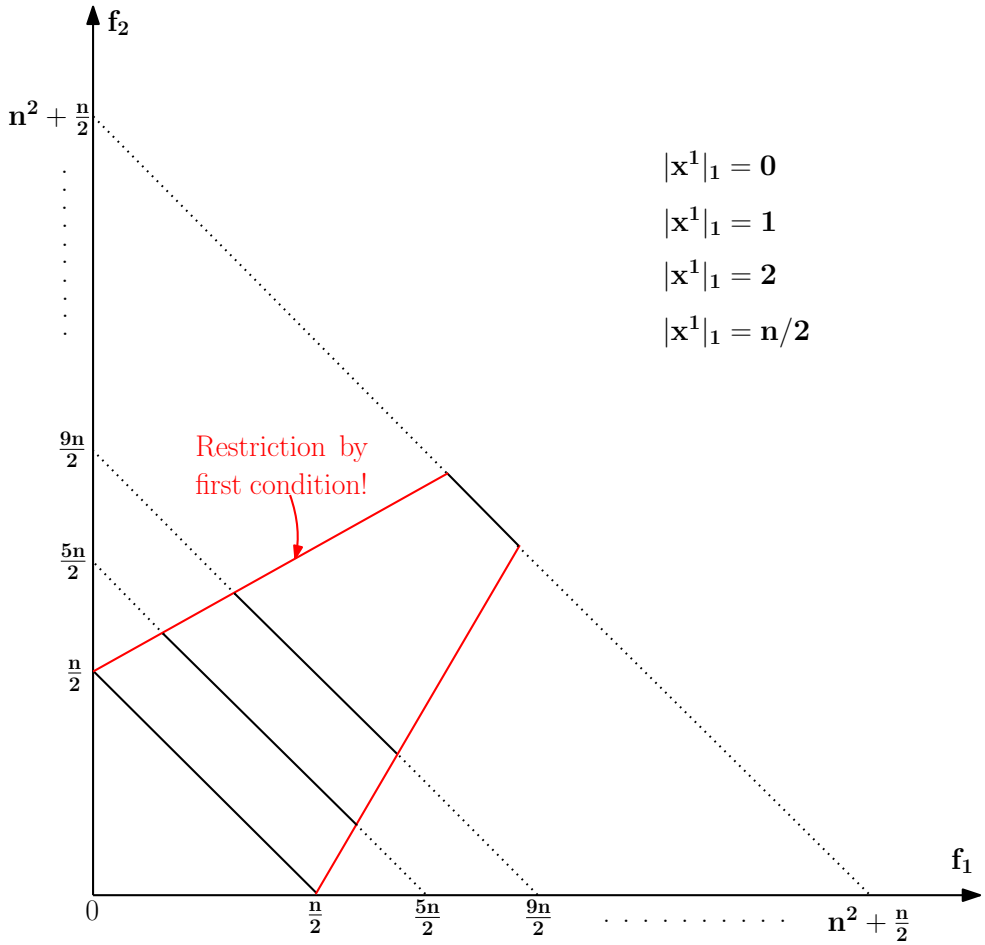

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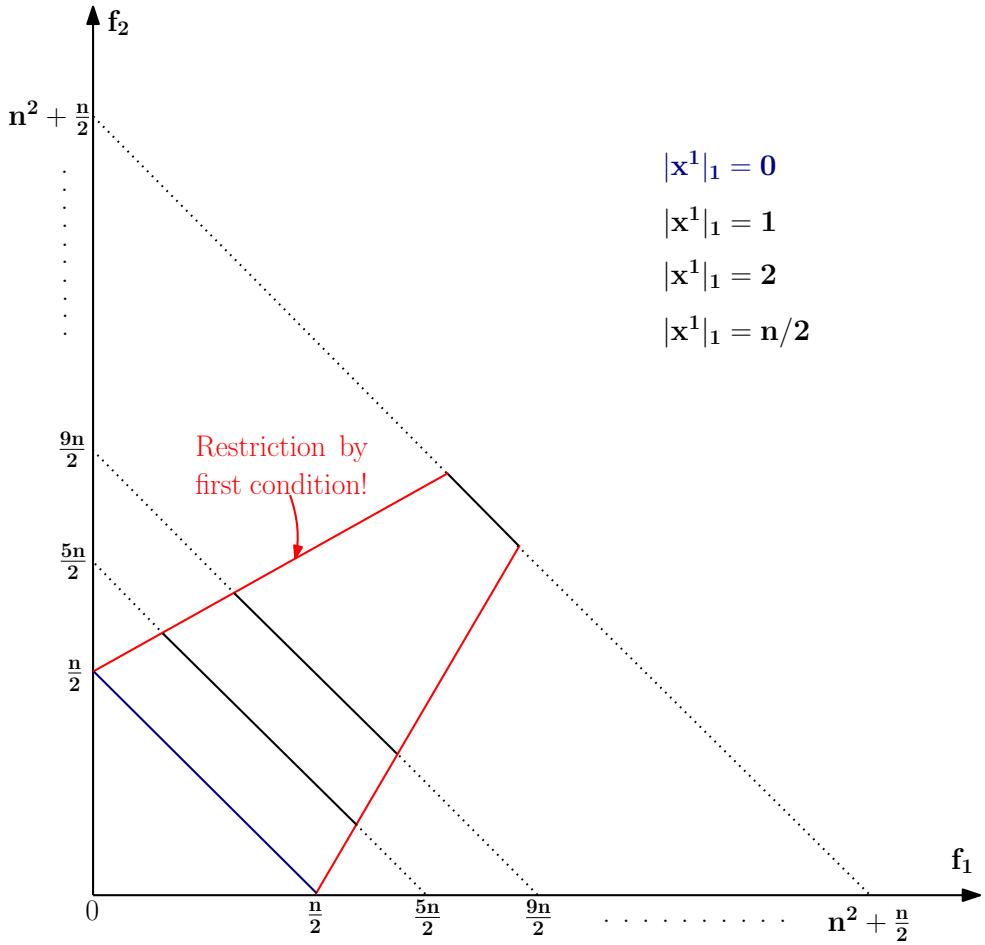

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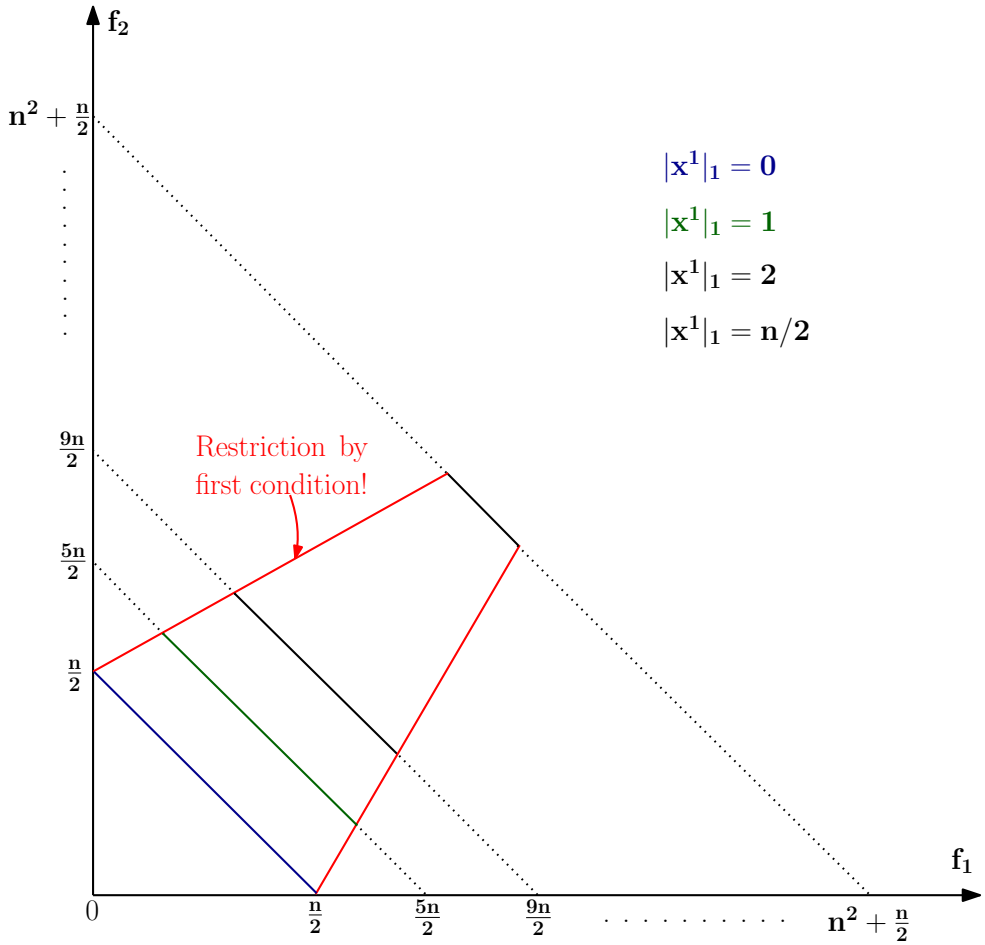

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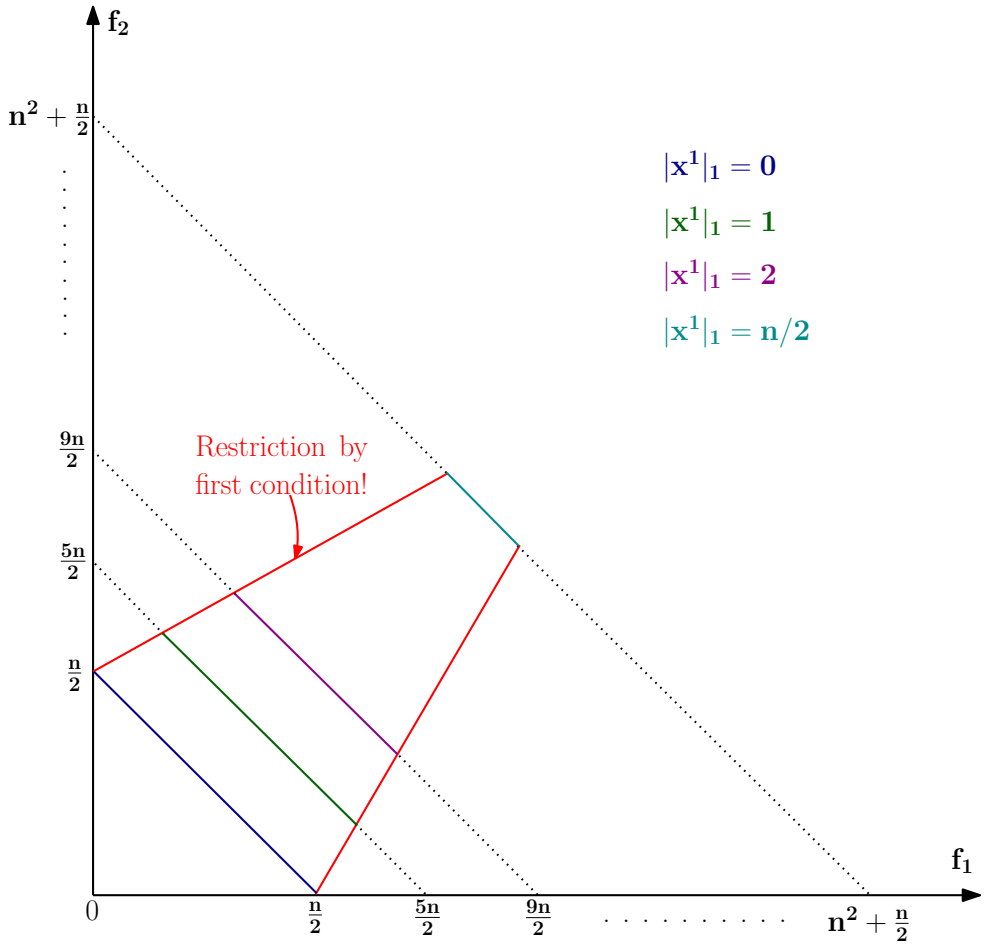

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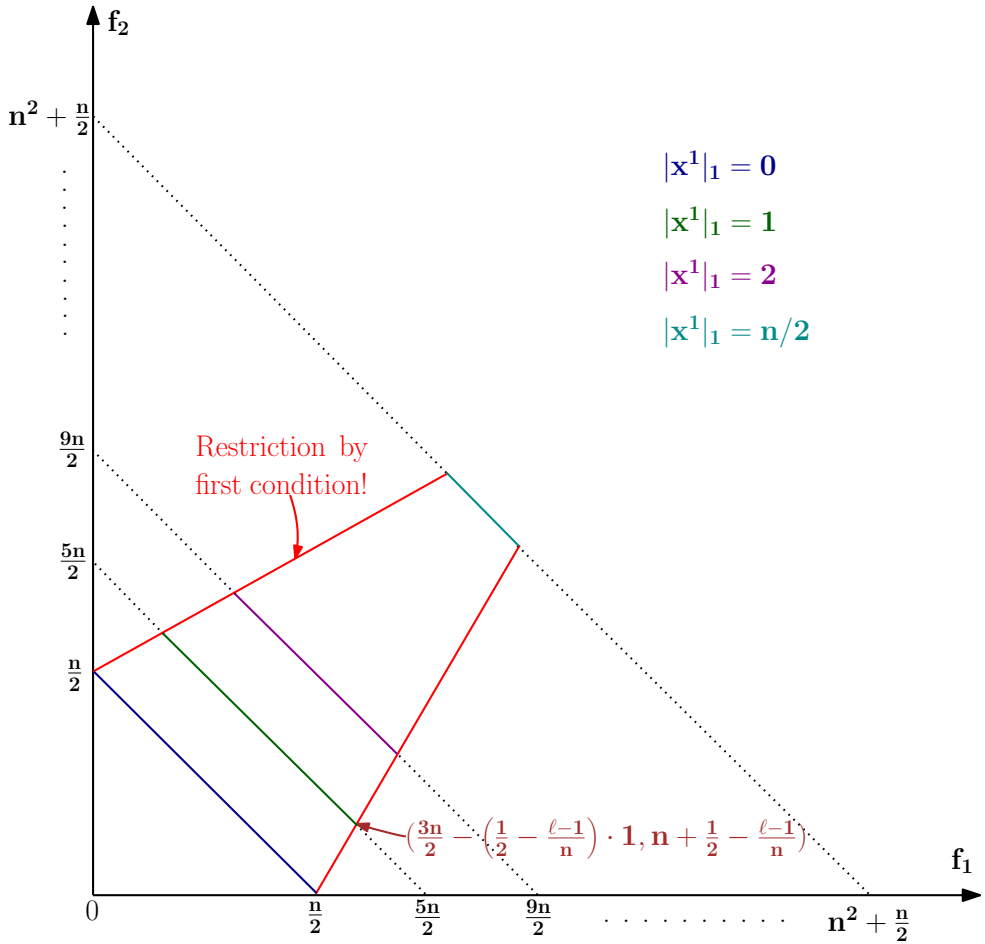

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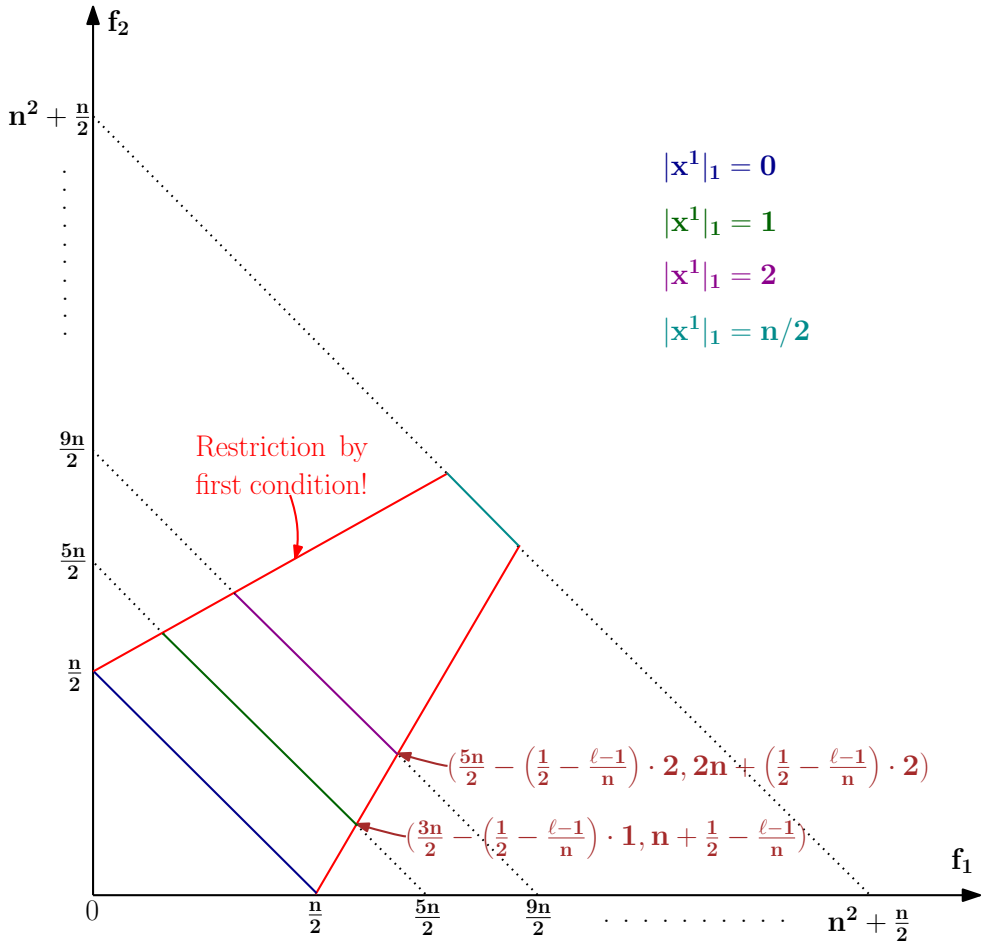

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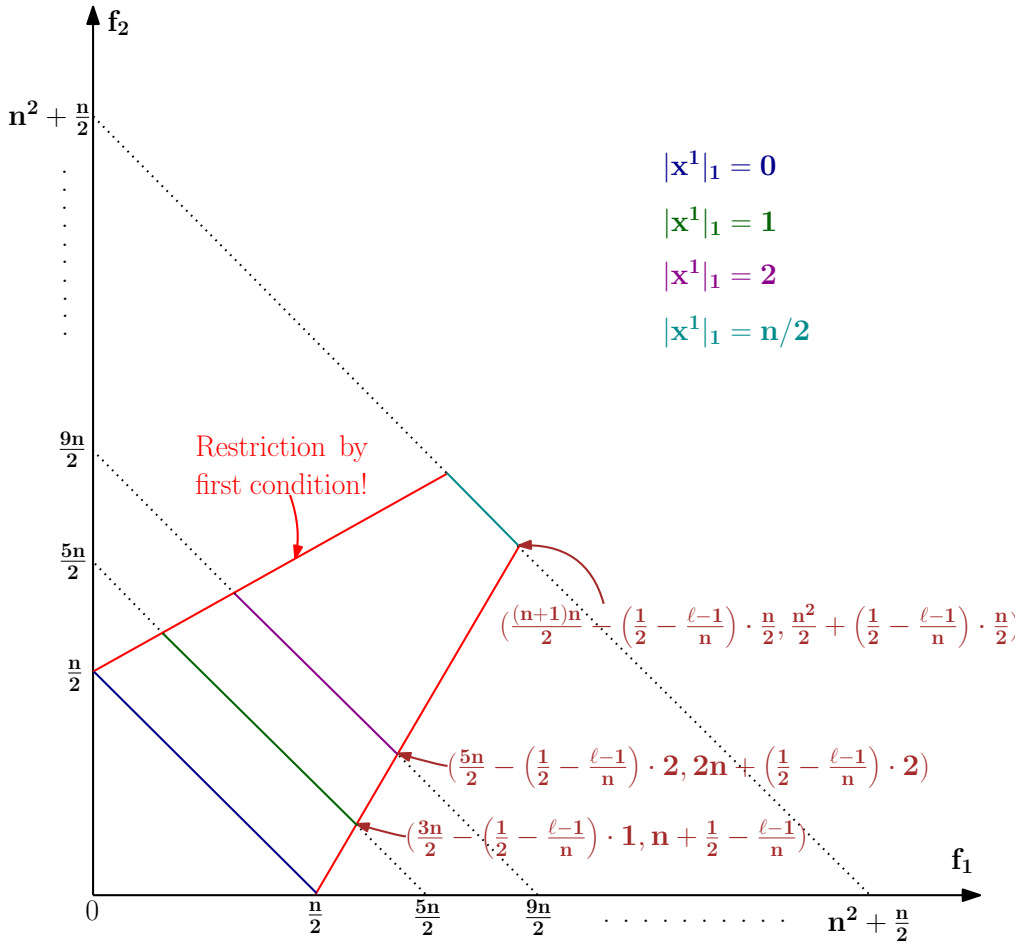

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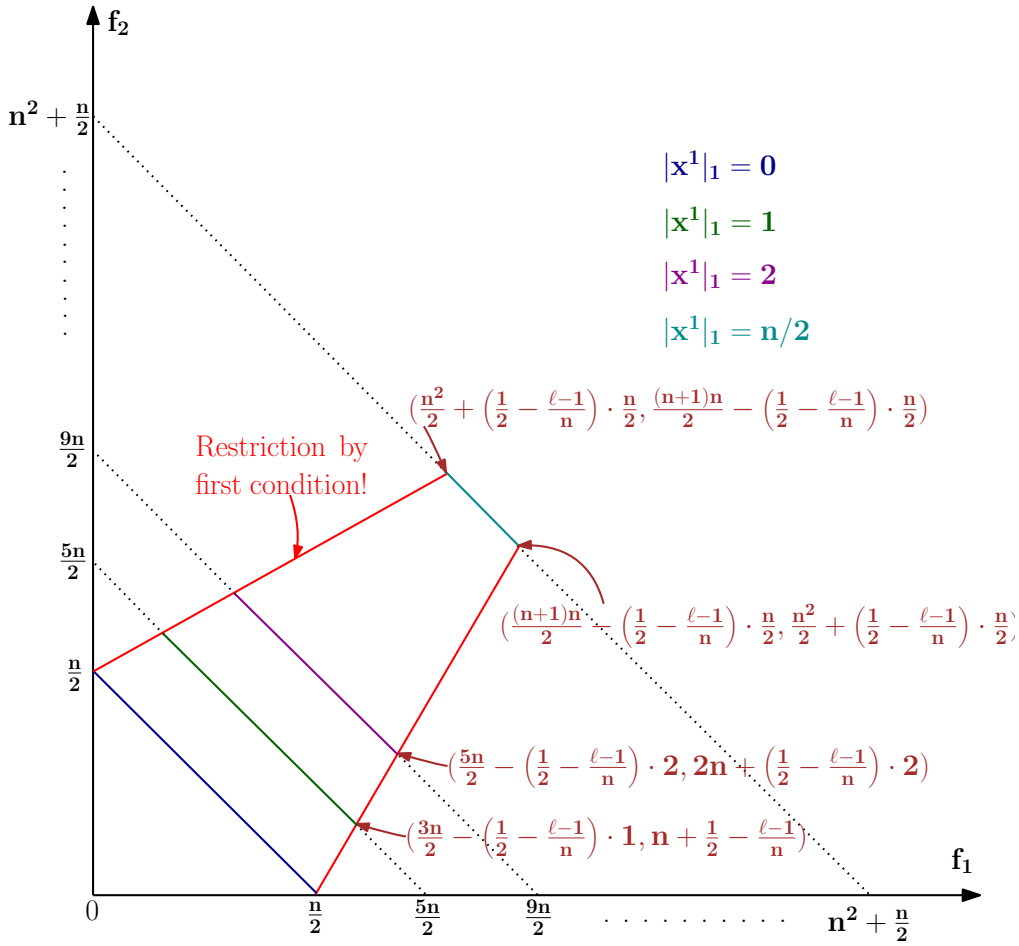

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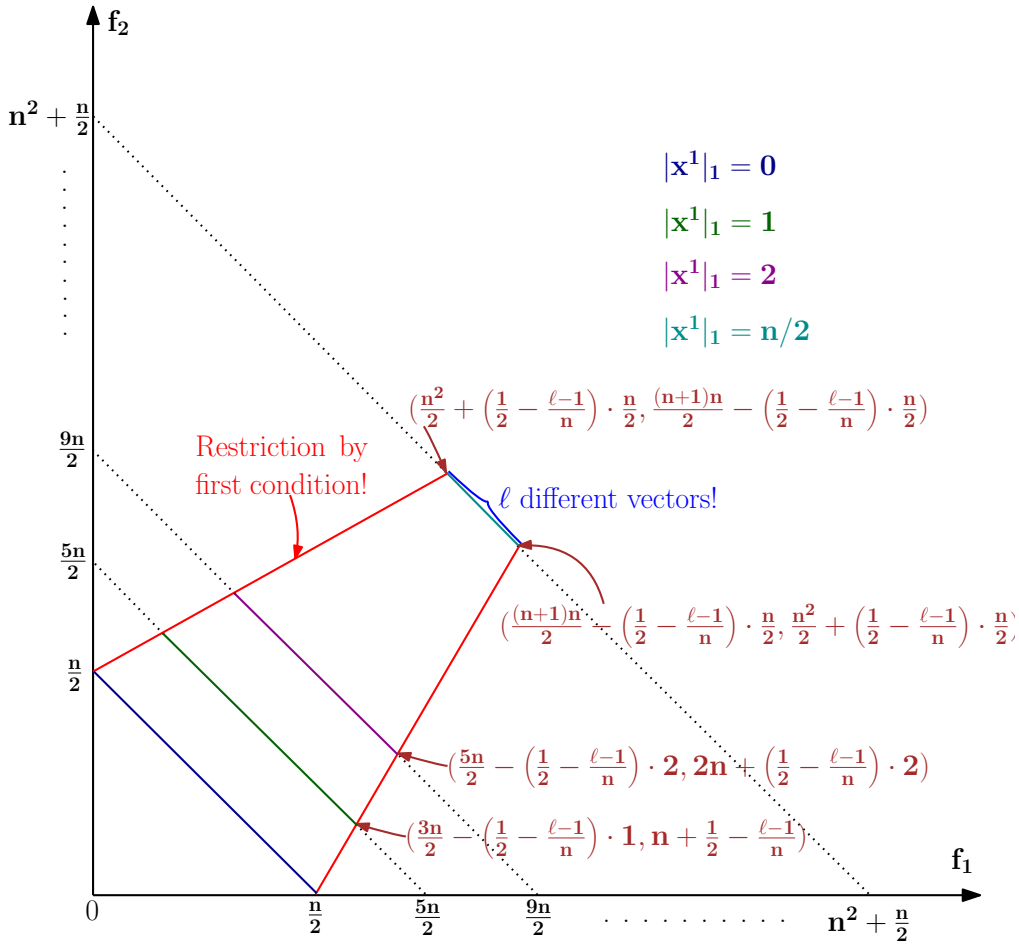

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Intuition: There is a positive drift of x towards $|x^2|_1 = n/2$

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$$\begin{cases} n \cdot |x^1|_1 \vec{1}_2 + (|x^2|_1, |x^2|_0), & \text{if } (\frac{1}{2} - \frac{\ell-1}{n})|x^1|_1 \leq |x^2|_1 \leq \frac{n}{2} - (\frac{1}{2} - \frac{\ell-1}{n})|x^1|_1, \\ (0,0), & \text{else.} \end{cases}$$

Formally: Define $W := \{x \in \{0,1\}^n \mid n/4 - \sqrt{n} \leq |x^2|_1 \leq n/4 + \sqrt{n}\}$

With the **additive drift theorem**, we can show the following:

Lemma

Suppose that there is $x \in P_t$ with $f(x) = 0$. Then, after $O(n\sqrt{n})$ iterations in expectation, $P_t \subset W$. So after $O(n\sqrt{n} \cdot n^{\sqrt{n}}) = e^{o(n)}$ iterations, there is $x \in P_t$ with $f(x) \neq 0$.

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Straightforwardly generalizable for $(\mu + \lambda)$-schemes!

Runtime Analysis of GSEMO and NSGA-II on CLIMB$_\ell$

Goal: Show that GSEMO outperforms NSGA-II by providing an upper bound of GSEMO and lower bound of NSGA-II on CLIMB$_\ell$

Show: GSEMO is by a superconstant factor faster than NSGA-II!

We start with an upper bound of GSEMO. The result is as follows

Theorem

Consider GSEMO optimizing $f := \text{CLIMB}_\ell$ for $\ell \in \{1, 3, \dots, n/2 + 1\}$. Then GSEMO needs at most $O(n \ln(n) + \ell^2 \ln(en/(n - 2\ell + 3)))$ fitness evaluations in expectation to cover the Pareto front of f.

Proof: Three phases:

- **1. Phase**: Create a solution with non-zero fitness: $1 + o(1)$ ✓
- **2. Phase**: Create a Pareto-optimal solution via hill-climbing with respect to the first half
- **3. Phase**: Cover the whole Pareto front

GSEMO on CLIMB$_\ell$ - 2.Phase: Hill-Climbing

Let $\ell \in \{1, 3, \ldots, n/2 + 1\}$ be odd. Then $f(x) := \text{CLIMB}_\ell(x) =$

$$\begin{cases} n \cdot |x^1|_1 \vec{1}_2 + (|x^2|_1, |x^2|_0), & \text{if } (\frac{1}{2} - \frac{\ell-1}{n})|x^1|_1 \leq |x^2|_1 \leq \frac{n}{2} - (\frac{1}{2} - \frac{\ell-1}{n})|x^1|_1, \\ (0,0), & \text{else.} \end{cases}$$

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The expected number of iterations GSEMO needs to find a Pareto optimal search point of f is $O(n \log(n))$

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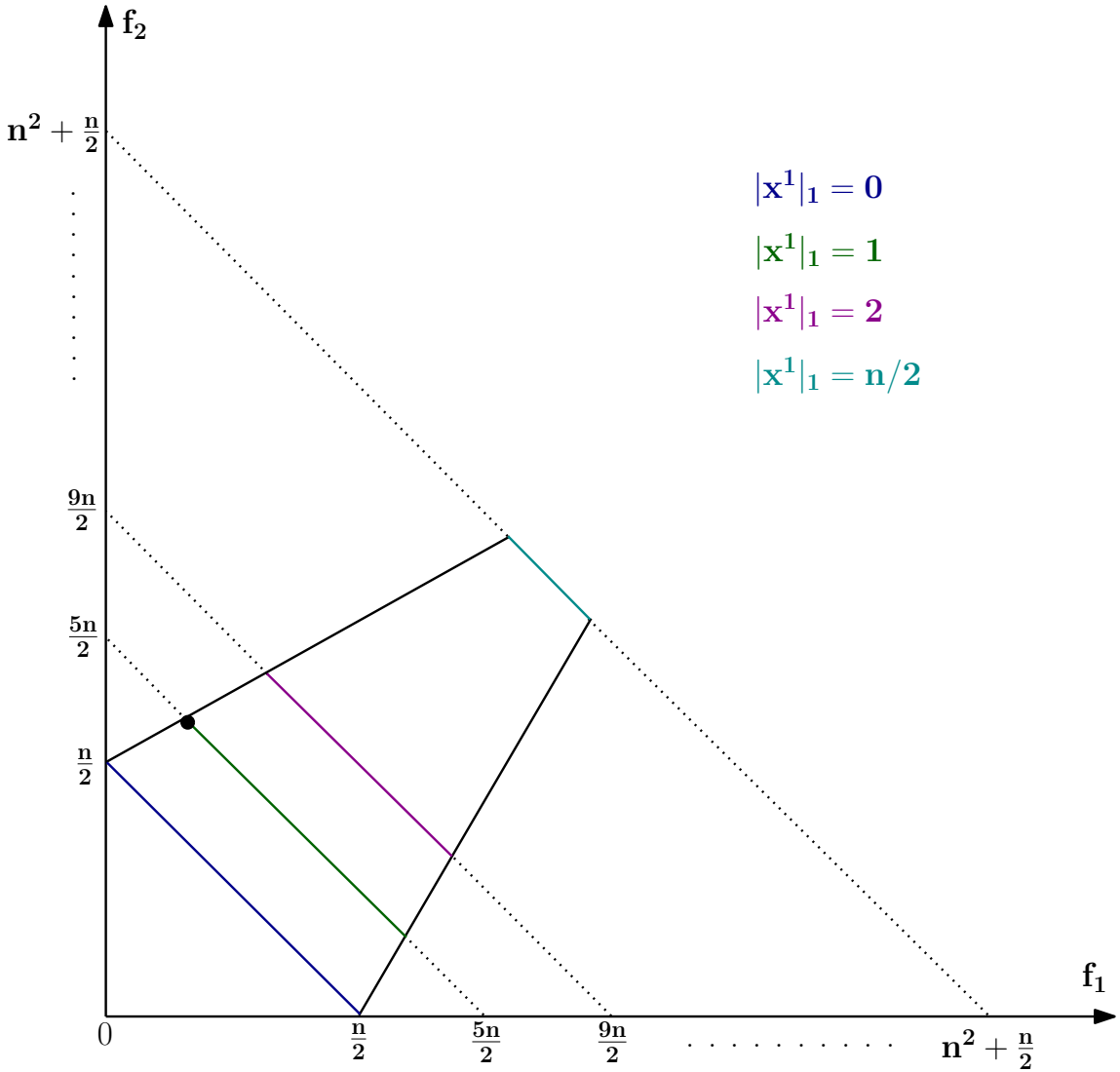

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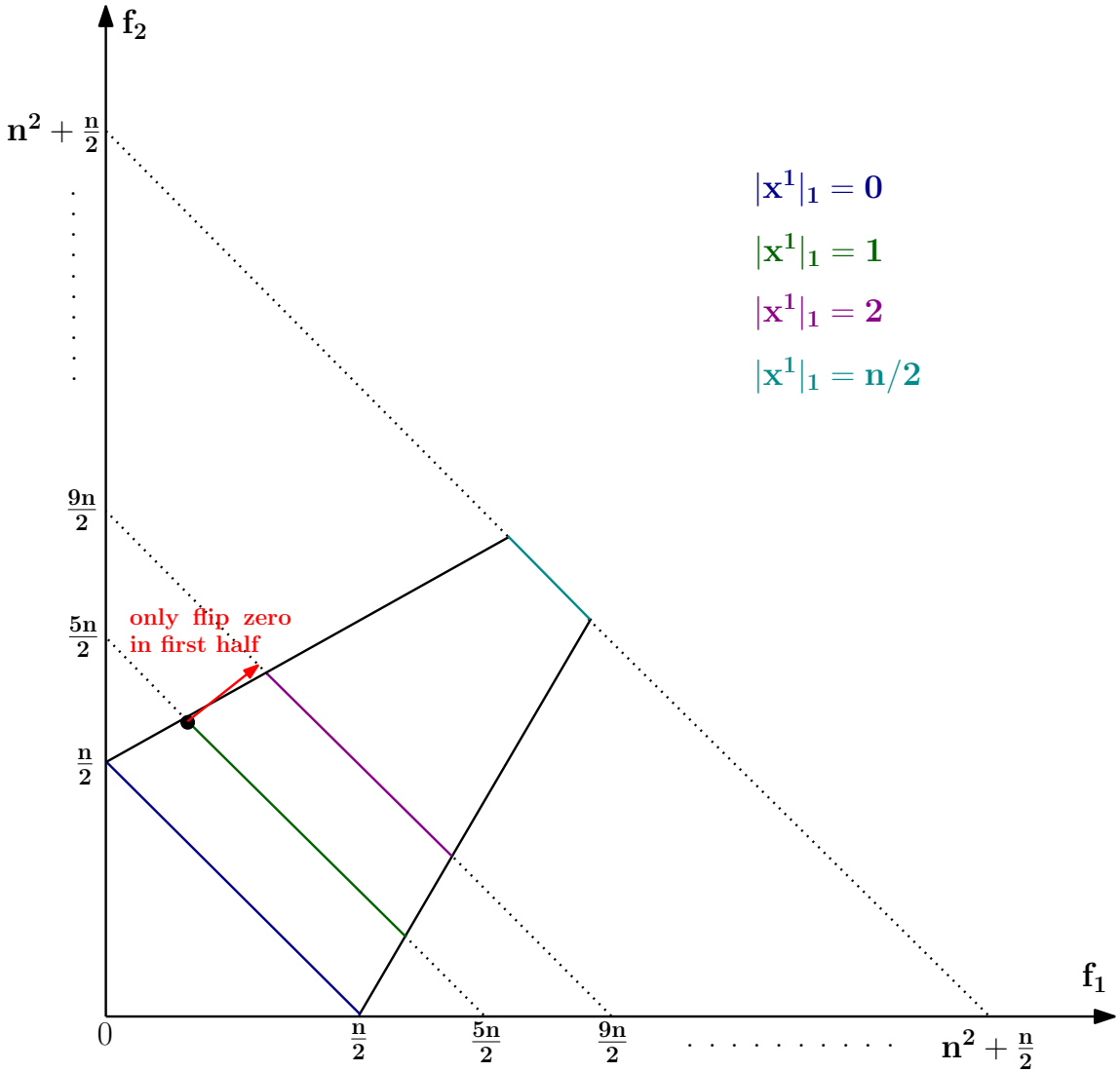

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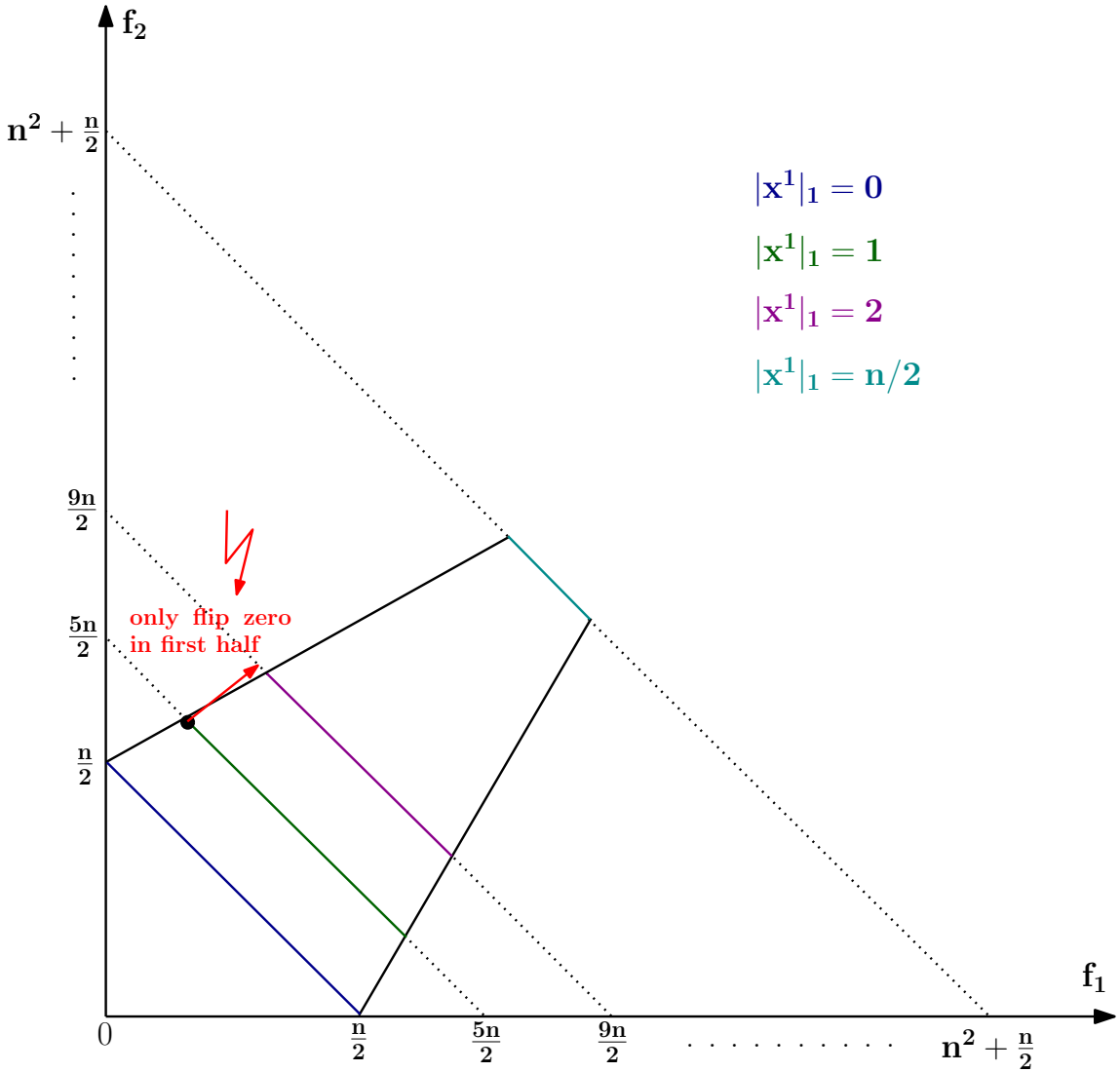

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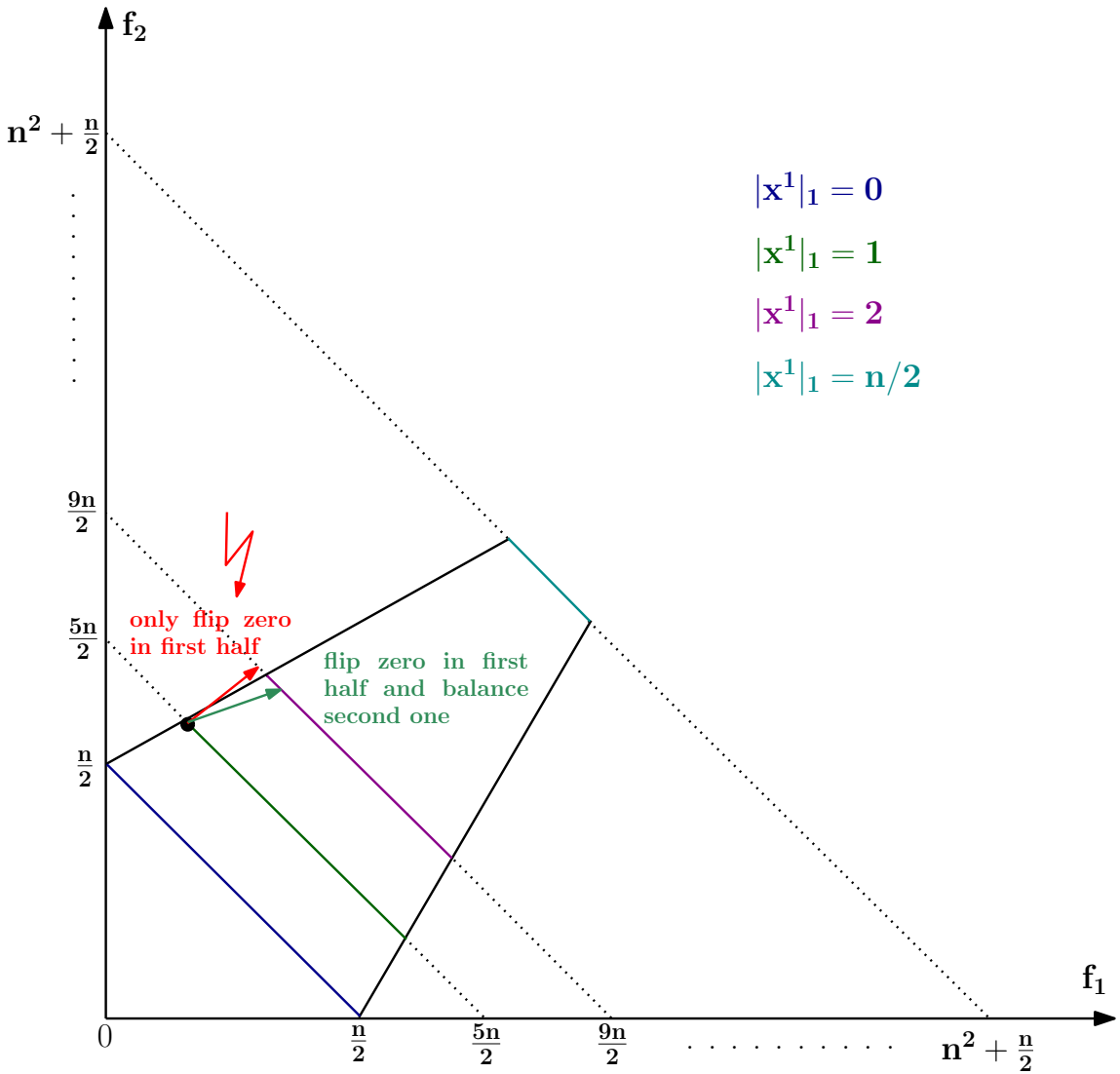

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- Expected number of generations: $\sum_{\ell=0}^{n/2-1} \frac{4en}{n/2-\ell} = O(n \log(n))$

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Consider GSEMO optimizing $f := \text{CLIMB}_\ell$ for $\ell \in \{1, 3, \dots, n/2 + 1\}$. Then GSEMO needs at most $O(n \ln(n) + \ell^2 \ln(en/(n - 2\ell + 3)))$ fitness evaluations in expectation to cover the Pareto front of f.

Proof: Three phases:

- **1. Phase**: Initialize the solution correctly: $1 + o(1)$ ✓
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Runtime Analysis of GSEMO and NSGA-II on CLIMB$_\ell$

Goal: Show that GSEMO outperforms NSGA-II by providing an upper bound of GSEMO and lower bound of NSGA-II on CLIMB$_\ell$

Show: GSEMO is by a superconstant factor faster than NSGA-II!

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A lower bound of NSGA-II on CLIMB$_\ell$

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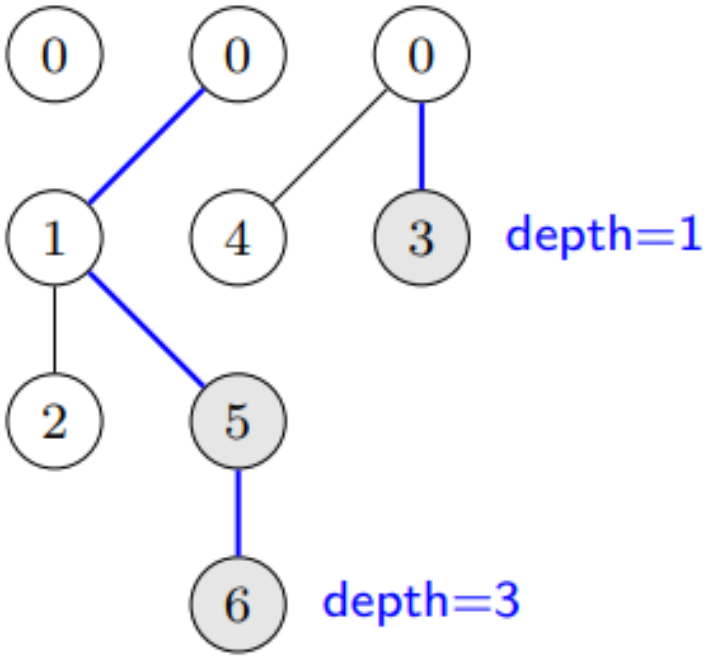

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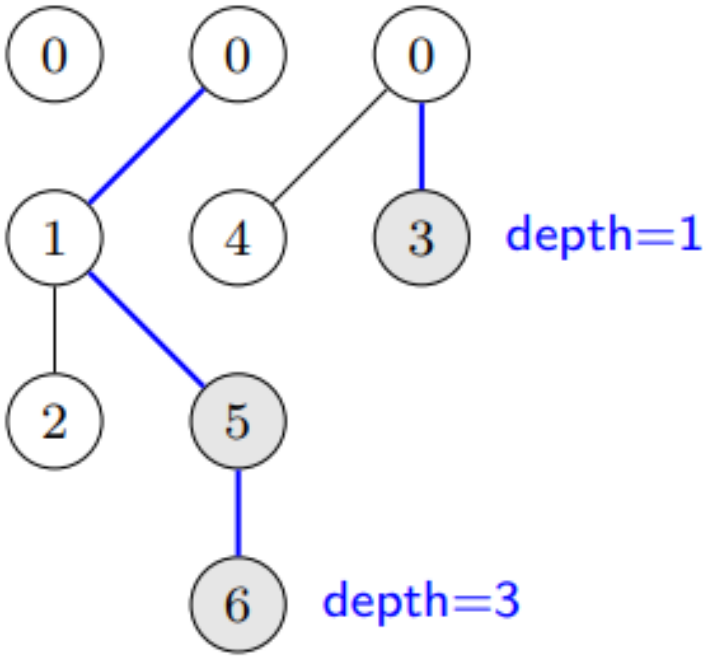

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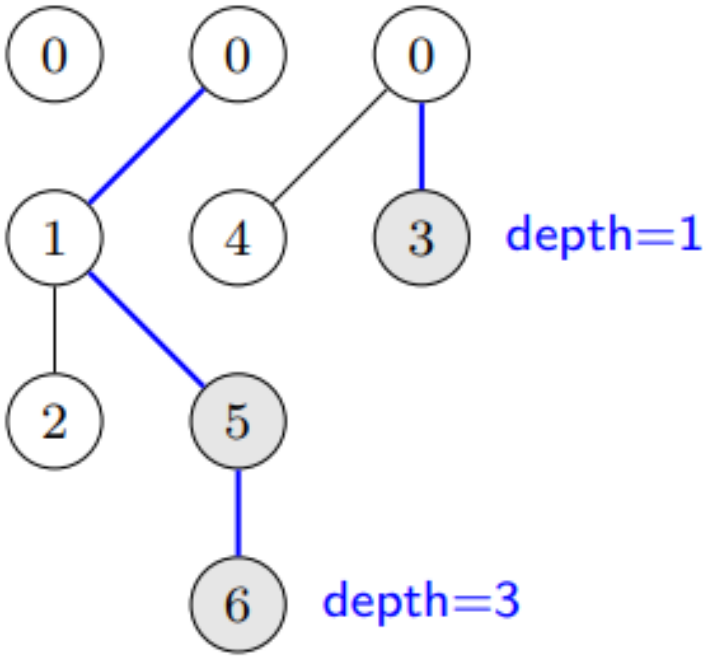
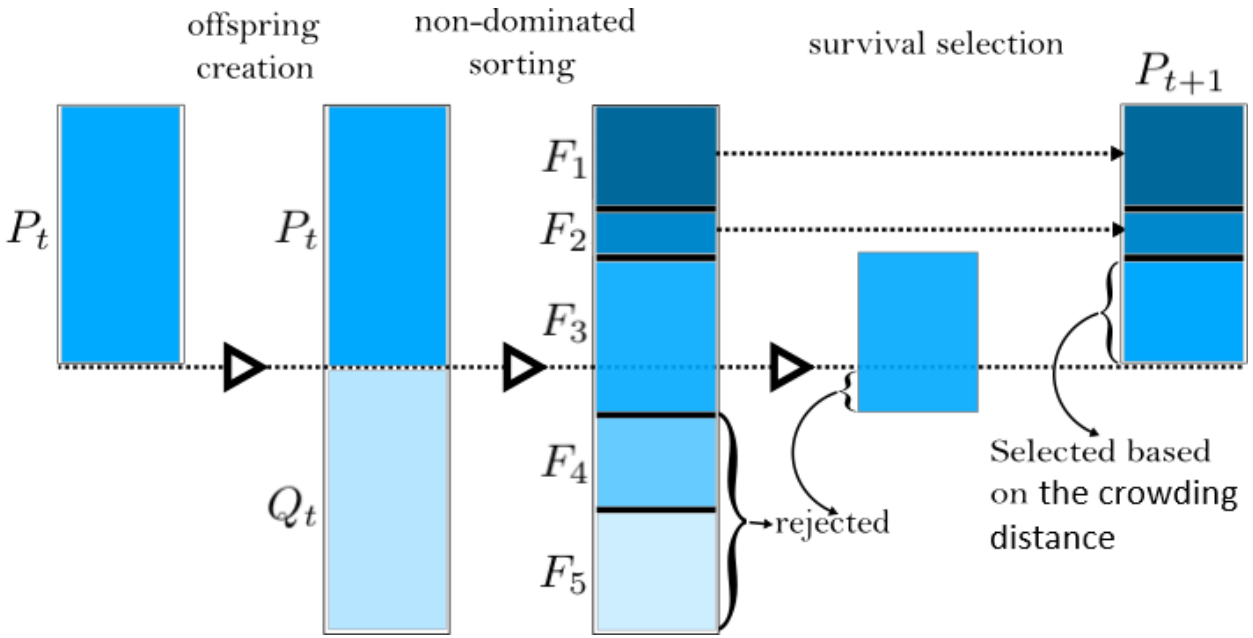


□

How to fix this difference in runtime?

Fix - The NSGA-II-DYN Algorithm



NSGA-II-DYN algorithm

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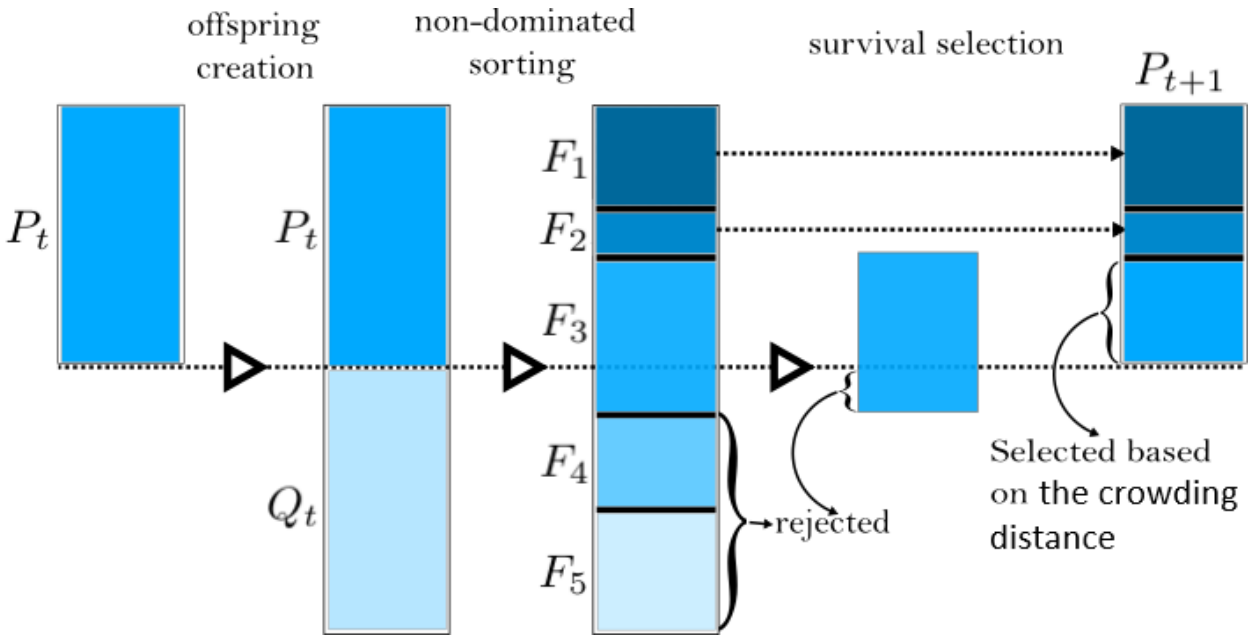

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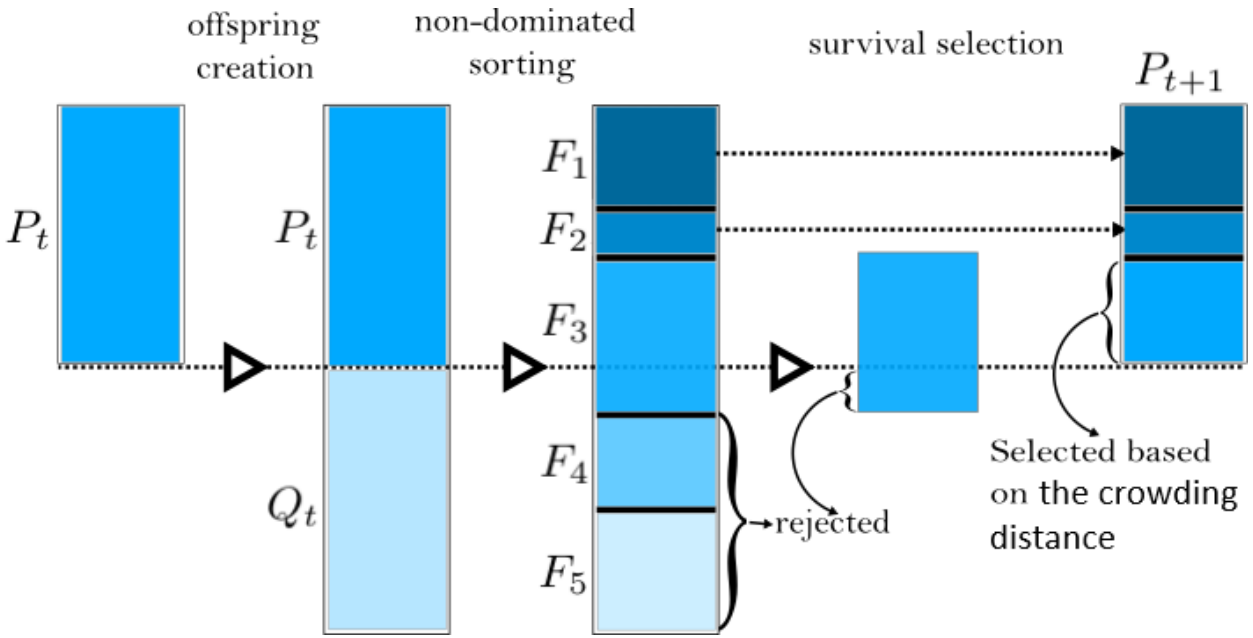

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- In each iteration, determine a maximum set of mutually incomparable, non-dominated solutions $S_t \subset P_t \cup Q_t$

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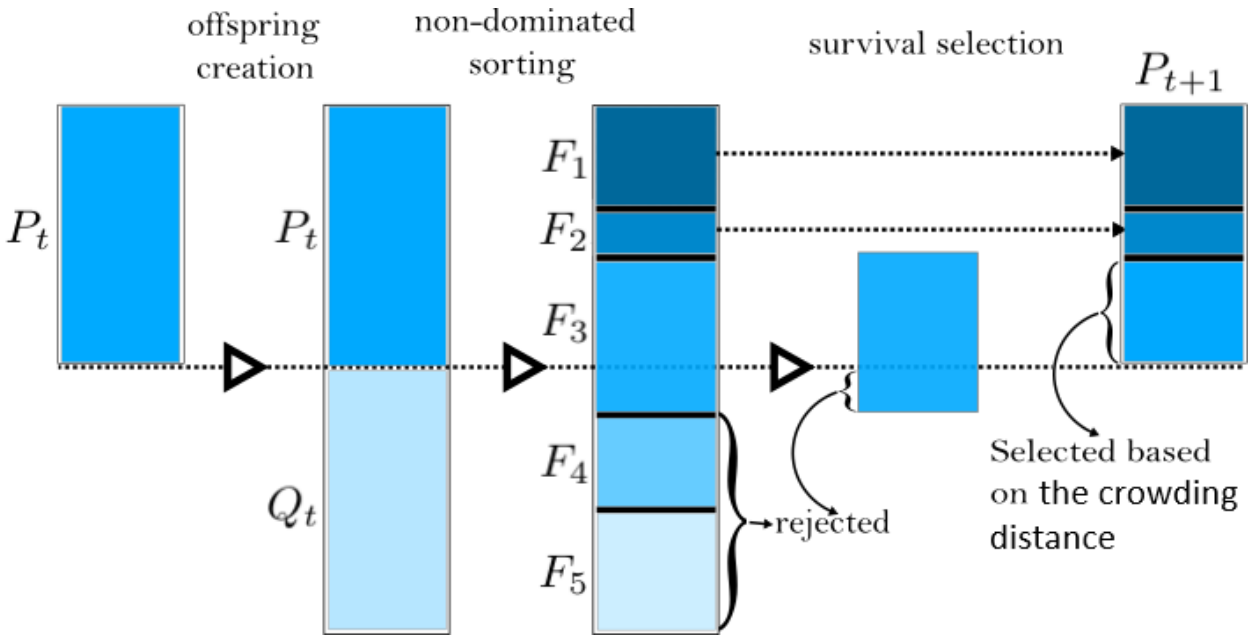

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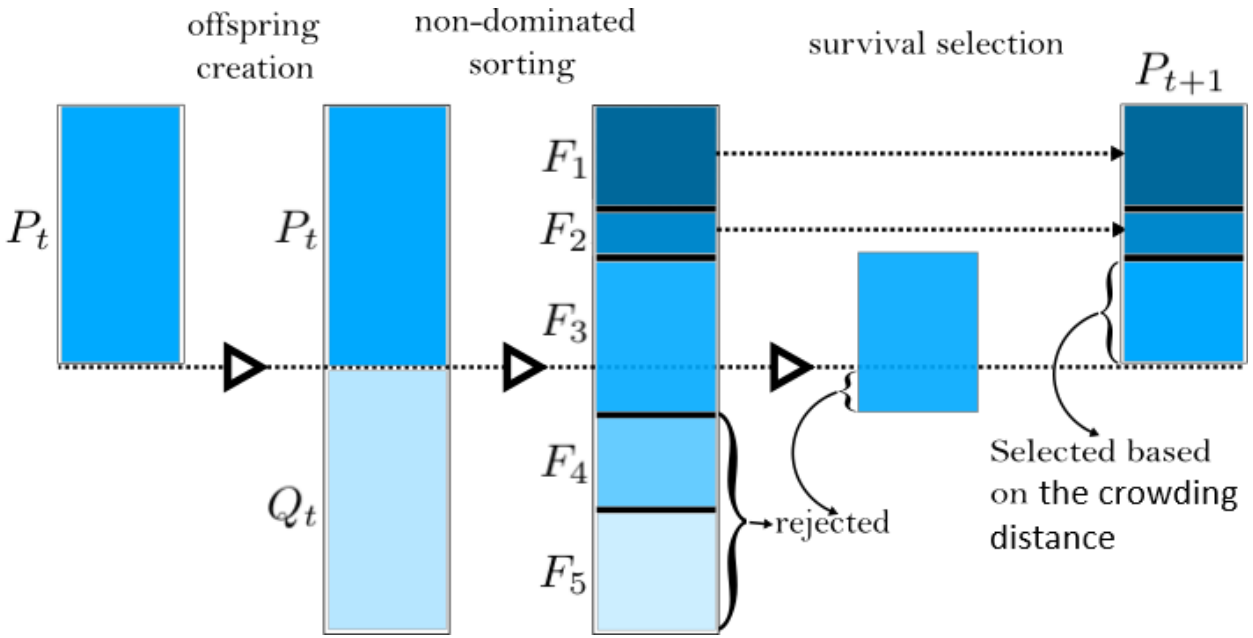

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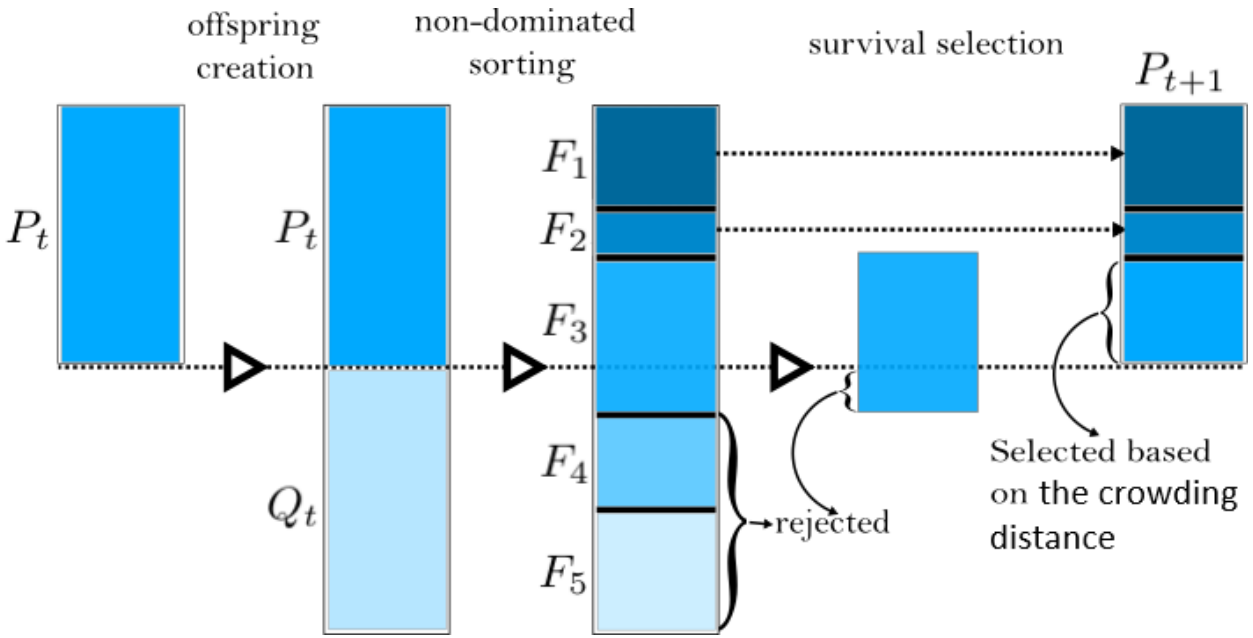

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- **Rank-one solutions are protected!**

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Crucial point: Control the population size of NSGA-II-DYN during hill-climbing

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Inductively: From $E[\mu_{t'}] \leq \frac{320}{n/2-(|x^1|_1-1)}$ conclude $E(\mu_{t^*}) \leq \frac{320}{n/2-|x^1|_1}$

Runtime Analysis of NSGA-II-DYN on CLIMB$_\ell$

Let $\ell \in \{1, 3, \ldots, n/2+1\}$ be odd. Then $f(x) := \text{CLIMB}_\ell(x) =$

$$\begin{cases} n \cdot |x^1|_1 \vec{1}_2 + (|x^2|_1, |x^2|_0), & \text{if } (\frac{1}{2} - \frac{\ell-1}{n})|x^1|_1 \leq |x^2|_1 \leq \frac{n}{2} - (\frac{1}{2} - \frac{\ell-1}{n})|x^1|_1, \\ (0,0), & \text{else.} \end{cases}$$

Goal: Prove that NSGA-II-DYN has asymptotically the same runtime (in terms of fitness evaluations) as GSEMO

Theorem

NSGA-II-DYN needs at most $O(n \ln(n) + \ell^2 \ln(en/(n-2\ell+3)))$ fitness evaluations in expectation to cover the Pareto front of f.

Proof: Three phases:

- **1. Phase**: Create a solution with non-zero fitness: $1 + o(1)$ ✓
- **2. Phase**: Create a Pareto-optimal solution via hill-climbing with respect to the first half: $O(n \log(n))$ ✓
- **3. Phase**: Cover the whole Pareto front

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Hope:

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- May guide the development of improved variants of our proposed NSGA-II-DYN

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