Physics-Informed Functional Tucker Method with RKHS Factors for Sparse Spatiotemporal Reconstruction

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01 / Sparse spatiotemporal reconstruction

Unknown: an evolving continuous field (e.g., velocity, temperature, vorticity)

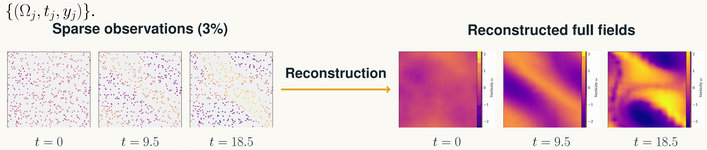
$$u : \mathcal{D} \times [0,T] \longrightarrow \mathbb{R}^C, \qquad \mathcal{D} \subset \mathbb{R}^d, \quad d \in \{2,3\}.$$

Given: sparse measurements at support times t_j:

$$y_j = \mathcal{P}_{\Omega_j} u(\cdot, t_j)\ (+\varepsilon_j), \qquad |\Omega_j| \ll N_{\text{sp}}.$$

Here Ω_j denotes sensor locations, N_{sp} the spatial grid size, and ε_j optional noise. Sensors are fixed per trajectory; entire frames may be missing.

Goal: recover $\hat{u}(\mathbf{x}, t)$ at unobserved spatial locations and missing timestamps within the support-time interval from $\{(\Omega_j, t_j, y_j)\}$.

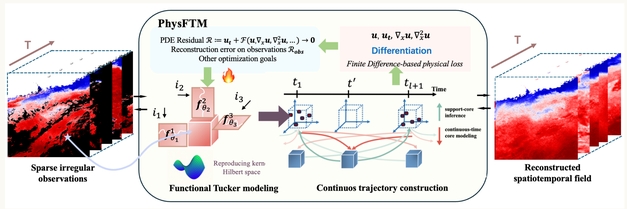


NS2D vorticity reconstruction at 3% sensors; three support times.

Challenges:

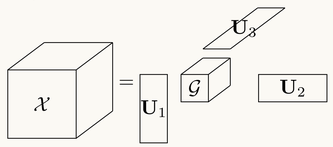
- **Severe undersampling:** many fields can fit the same sparse measurements.
- **Continuous reconstruction:** recover off-sensor structures and dynamics between support frames.
- **Physical consistency:** maintain agreement with the governing PDEs beyond observed sensors.

02/ PhysFTM: an overview



- **Functional Tucker model**

Compact Tucker cores and continuous RKHS factors represent the spatial field.



$\mathcal{G}(t)$: **core tensor**, depending on time t.

U_k: **factor matrices**, representing space through RKHS functions shared across time.

- **Optimization**

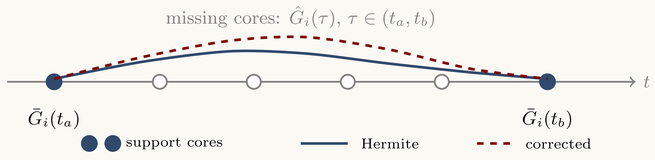
Infer cores by least squares; update RKHS factors under data and physics constraints.

$$\mathcal{L}_{\text{basis}} = \underbrace{\mathcal{L}_1}_{\text{Reconstruction loss}} + \underbrace{\mathcal{L}_2}_{\text{Spatial regularity}} + \underbrace{\mathcal{L}_3}_{\text{PDE consistency}} + \underbrace{\mathcal{L}_4}_{\text{Temporal regularity}}$$

Weighted terms: observation, RKHS, PDE and time-kernel losses.

- **Continuous interpolation**

Fit support cores in continuous time, then decode missing-time fields.



03 / Functional Tucker model

From factor matrices to continuous spatial functions

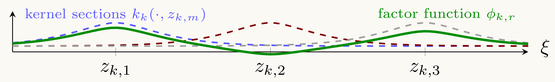
$$\hat{u}(\mathbf{x}, t, c) = \sum_{r_1=1}^{R_1} \cdots \sum_{r_d=1}^{R_d} G_{r_1,\dots,r_d,t,c} \prod_{k=1}^{d} \phi_{k,r_k}(x_k) = \mathcal{D}_\Phi(G(t))(\mathbf{x}, c).$$

$d \in \{2,3\}$; $\mathbf{x} = (x_1, \dots, x_d)$; $(R_1, R_2, \dots, R_d)$ Tucker ranks.

$G(t) \in \mathbb{R}^{R_1 \times \cdots \times R_d \times C}$; the spatial factors Φ are shared across times and channels.

RKHS factors: a finite kernel expansion

$$\phi_{k,r}(\xi) = \sum_{m=1}^{M_k} a_{k,m,r}\, k_k(\xi, z_{k,m}), \qquad A_k \in \mathbb{R}^{M_k \times R_k}.$$



$z_{k,m}$ are inducing points; the kernel defines the function space and A_k learns the factor functions.

04 / An alternating optimization scheme

1. Fix the factors; solve the support-time cores

$$\min_{\{G_t\}} \sum_{t=1}^{T_s} \|M_t \odot (\Phi\, \text{vec}(G_t) - Y_t)\|_2^2 + \lambda_{\text{ridge}} \sum_t \|G_t\|_F^2 + \beta_{\text{TV}} \sum_{t=2}^{T_s} \|G_t - G_{t-1}\|_F^2.$$

A quadratic least-squares problem solved by conjugate gradient method.

2. Fix the cores; update RKHS coefficients with AdamW

$$\mathcal{L}_{\text{basis}} = \mathcal{L}_{\text{obs}} + \lambda_{\text{RKHS}}\mathcal{L}_{\text{RKHS}} + \lambda_{\text{PDE}}\mathcal{L}_{\text{PDE}} + \lambda_t \mathcal{L}_{\text{time}}.$$

$$\mathcal{L}_{\text{obs}} = \frac{1}{|\Omega|} \sum_{\alpha \in \Omega} (\hat{Y}_\alpha - Y_\alpha)^2. \qquad \mathcal{L}_{\text{RKHS}} = \sum_{k=1}^{d} \text{tr}(A_k^\top K_k A_k).$$

$$\mathcal{L}_{\text{PDE}} = \frac{1}{N_b |\mathcal{Q}| |T'|} \sum_{b=1}^{N_b} \sum_{\mathbf{x} \in \mathcal{Q}} \sum_{t \in \mathcal{T}'} \|\mathcal{R}_h[\hat{u}_b](\mathbf{x}, t)\|_2^2. \qquad \mathcal{L}_{\text{time}} = \frac{1}{D_G} \sum_{b=1}^{N_b} \text{tr}((G_b^\flat)^\top K_t^{-1} G_b^\flat).$$

RKHS norms control spatial complexity; K_t is the RBF time kernel; finite-difference PDE residuals supervise off-sensor points. $D_G = C \prod_k R_k$. $\mathcal{L}_{\text{time}}$ is constant for fixed cores; ridge/TV penalties belong to the core solve.

05 / Continuous interpolation

Infer support cores under the frozen spatial decoder

$$\bar{G}_t = G_t^{\text{ref}} + \Delta G_t^\star, \qquad G_t^{\text{ref}} = \frac{1}{N_{\text{train}}} \sum_{i=1}^{N_{\text{train}}} G_{i,t}^{\text{train}}.$$

Solve for a correction around the mean of the trained cores.

Fit a continuous trajectory in core space

$$\hat{G}(\tau) = \underbrace{B(\tau)}_{\text{Hermite baseline}} + \underbrace{\gamma(\tau)\Delta_\psi(\tau)}_{\text{ODE–RNN correction}}, \qquad \gamma(\tau) = \sin^2\!\left(\pi \frac{\tau - t_a}{t_b - t_a}\right).$$

$$\mathcal{L}_{\text{fit}} = \frac{1}{|\mathcal{T}_{\text{miss}}| D_G} \sum_{\tau \in \mathcal{T}_{\text{miss}}} \|\hat{G}(\tau) - G^{\text{train}}(\tau)\|_F^2.$$

Since $\gamma(t_a) = \gamma(t_b) = 0$, the inferred support cores are preserved.

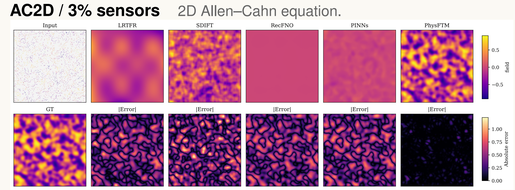
Decode at the requested spatial coordinate and time

$$\hat{u}(\mathbf{x}, \tau, c) = \mathcal{D}_\Phi(\hat{G}(\tau))(\mathbf{x}, c), \qquad \tau \in [t_1, t_{T_s}].$$

Spatial factors stay fixed; only the Tucker core varies with time.

06 / Comparison with other methods on 2D PDEs

AC2D / **3% sensors** 2D Allen–Cahn equation.

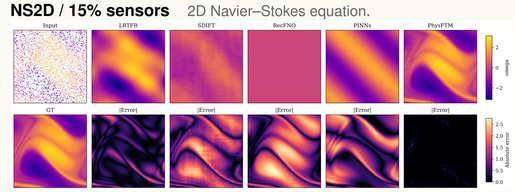


$$u_t = \lambda(\epsilon^2 \Delta u - u^3 + u).$$

$\lambda = 1$, $\epsilon = 0.01$; periodic domain.

Result
PhysFTM recovers fine-scale patterns with the lowest off-sensor RMSE.

NS2D / **15% sensors** 2D Navier–Stokes equation.



$$\omega_t + \mathbf{v} \cdot \nabla \omega = \nu \Delta \omega + f.$$

$\mathbf{v} = (u, v)$, $\nu = 10^{-4}$; forced periodic flow.

Result
PhysFTM preserves vortical structures with the lowest off-sensor RMSE.

07 / Quantitative results

Method	Random-drop-half				Temporal-interval-missing			
	S-sup	Off-sup	S-miss	Off-miss	S-sup	Off-sup	S-miss	Off-miss
AC2D 2D Allen–Cahn equation								
PhysFTM	**0.01735**	**0.08710**	0.05035	**0.09988**	**0.01633**	**0.09397**	0.03953	**0.07924**
SDIFT	0.35804	0.38080	0.38030	0.38151	0.35699	0.36199	0.39071	0.39259
LRTFR	0.31170	0.32220	0.31340	0.32038	0.31006	0.31676	0.32059	0.32781
RecFNO	0.01848	0.45283	**0.02178**	0.45523	0.01811	0.45875	**0.02116**	0.43680
SIREN-MLP	0.34868	0.34932	0.35013	0.35076	0.34605	0.34664	0.35819	0.35898
PINNs	0.35166	0.35243	0.35268	0.35331	0.34874	0.34943	0.36508	0.36622
NS2D 2D Navier–Stokes equation								
PhysFTM	**0.00669**	**0.06210**	**0.03240**	**0.06498**	**0.00435**	**0.06282**	**0.14949**	**0.15056**
SDIFT	0.52867	0.50970	0.51003	0.51027	0.53019	0.53052	0.50434	0.50275
LRTFR	0.13245	0.14720	0.13447	0.14576	0.13515	0.14682	0.19025	0.19661
RecFNO	0.06810	0.09757	0.07820	0.10441	0.06906	0.09910	0.35803	0.35943
SIREN-MLP	0.24555	0.24526	0.24465	0.24441	0.24797	0.24777	0.21369	0.21354
PINNs	0.34254	0.34177	0.34101	0.34055	0.34722	0.34657	0.31892	0.31848
Heat3D 3D heat equation								
PhysFTM	**0.00108**	**0.00125**	**0.00575**	**0.00574**	**0.00122**	**0.00137**	**0.00310**	**0.00305**
SDIFT	0.05811	0.05808	0.06303	0.06272	0.06284	0.06268	0.05016	0.04989
LRTFR	0.01212	0.01222	0.01309	0.01315	0.01210	0.01225	0.00577	0.00577
RecFNO	0.28736	0.77847	0.26372	0.72034	0.28629	0.77526	0.22597	0.64187
SIREN-MLP	0.04080	0.04087	0.04211	0.04205	0.04342	0.04344	0.02749	0.02742
PINNs	0.04000	0.04011	0.04271	0.04264	0.04314	0.04319	0.03353	0.03334
Ocean3D 3D ocean currents								
PhysFTM	**0.00632**	**0.06574**	**0.07133**	**0.08858**	**0.00595**	**0.06382**	0.18985	0.18565
SDIFT	0.27863	0.28065	0.27585	0.27838	0.27494	0.27725	0.28712	0.28941
LRTFR	0.11038	0.11450	0.11767	0.12155	0.10506	0.10829	**0.18062**	**0.18377**
RecFNO	0.73827	0.97074	0.77688	0.96302	0.71841	0.94379	0.99191	1.05383
SIREN-MLP	0.20158	0.20168	0.20024	0.20094	0.19773	0.19833	0.21199	0.21151
PINNs	0.30163	0.30100	0.30232	0.30194	0.29765	0.29739	0.31870	0.31725

Reading the table

RMSE ↓
3% sensors; lower is better.

Bold: best.
Underline: second best.

Evaluation
S: sensor
Off: off-sensor
sup: support frames
miss: missing frames

Protocols
Random: retain half the internal frames on average.
Interval: withhold an internal 20% time block.

For more results

GitHub Code

08 / Ocean3D: evolution over time

3D ocean currents from Copernicus Marine GLORYS12V1 reanalysis. 15% sensors.

