NEURAL INFORMATION PROCESSING SYSTEMS

Complex Schrödinger Bridges

NeurIPS 2026

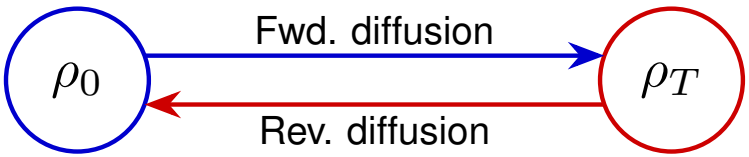
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IMPERIAL

Motivation

Diffusion Bridges: a class of bidirectional transportation models with diffusion.



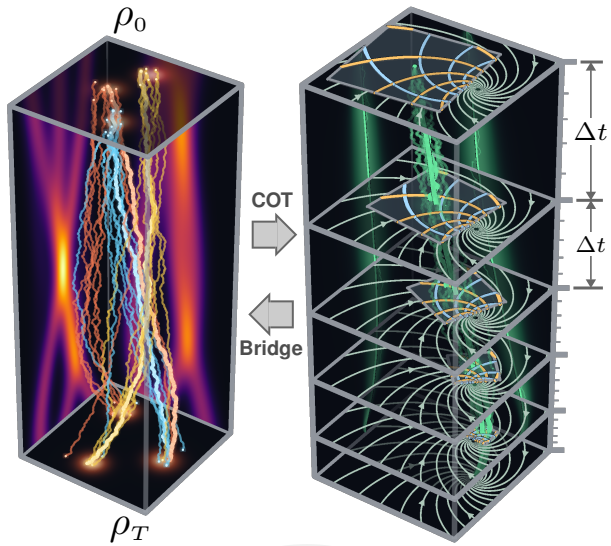
General Schrödinger Bridge.

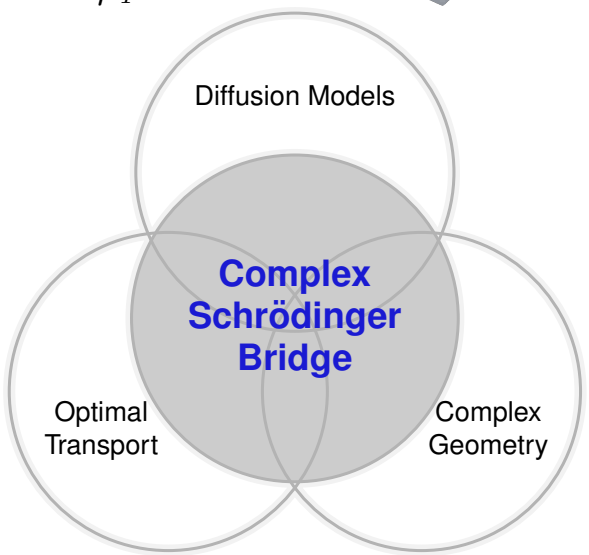
$$\mathrm{d}Z_t = i\,\nabla_\alpha H\Big(Z_t, \bar{Z}_t, \nabla_{\mathbf{z}}\psi_t, \nabla_{\bar{\mathbf{z}}}\psi_t, \rho_t\Big)\,\mathrm{d}t + \sigma\,\mathrm{d}W_t$$
$$i\frac{\partial \Psi}{\partial t} = \hat{H}\Psi \;\text{ and }\; i\frac{\partial \bar{\Psi}}{\partial t} = -\hat{H}^\dagger\bar{\Psi},$$

AI4Science: Usage of complex-valued data.

Question

What principles should govern the real-world dynamics of diffusion bridges?





Complex Langevin System

Generalized Langevin Diffusion

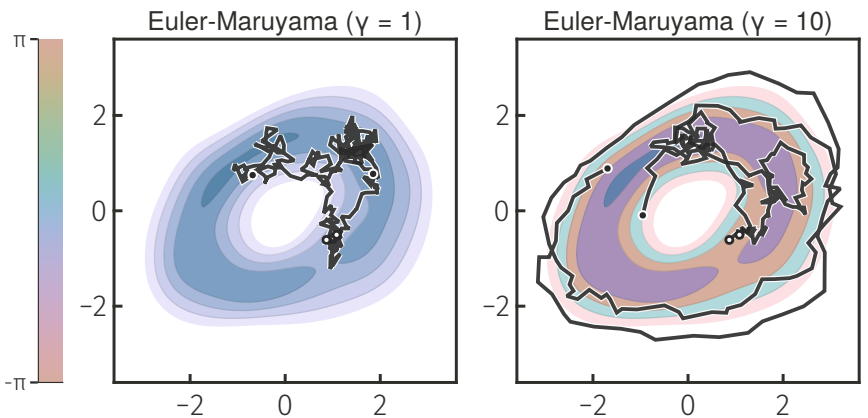
$$\begin{bmatrix} \mathrm{d}\mathbf{x}_t \\ \mathrm{d}\mathbf{v}_t \end{bmatrix} = \gamma \cdot \underbrace{\frac{\sigma^2}{4}\begin{pmatrix} -\nabla_{\mathbf{v}} \log p(\mathbf{x}_t, \mathbf{v}_t) \\ \nabla_{\mathbf{x}} \log p(\mathbf{x}_t, \mathbf{v}_t) \end{pmatrix}\mathrm{d}t}_{\text{Hamiltonian dynamics}} + \underbrace{\frac{\sigma^2}{4}\begin{pmatrix} \nabla_{\mathbf{x}} \log p(\mathbf{x}_t, \mathbf{v}_t) \\ \nabla_{\mathbf{v}} \log p(\mathbf{x}_t, \mathbf{v}_t) \end{pmatrix}\mathrm{d}t + \frac{\sigma}{\sqrt{2}}\begin{pmatrix} \mathrm{d}\mathbf{w}_t \\ \mathrm{d}\hat{\mathbf{w}}_t \end{pmatrix}}_{\text{Langevin diffusion}}$$

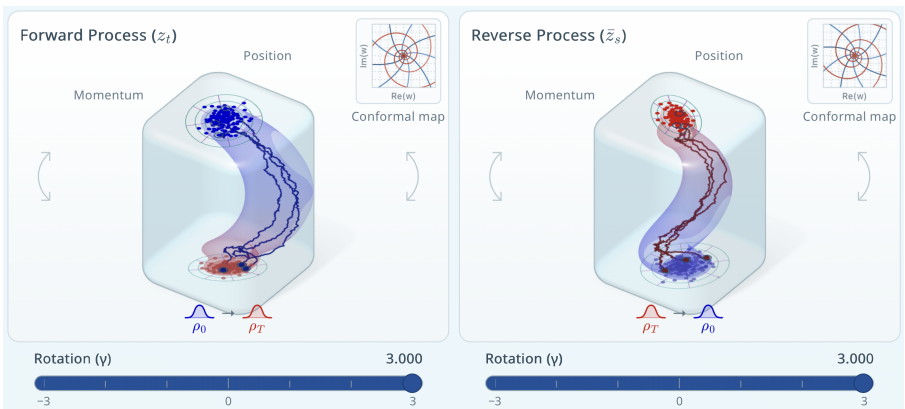
Definition 1 (Wirtinger operators).

$$\begin{bmatrix} \nabla_{\mathbf{z}} f \\ \nabla_{\bar{\mathbf{z}}} f \end{bmatrix} := \frac{1}{2}\left(\begin{pmatrix} 1 & -i \\ 1 & +i \end{pmatrix} \otimes \mathbf{I}_d\right)\begin{pmatrix} \nabla_{\mathbf{x}} f \\ \nabla_{\mathbf{v}} f \end{pmatrix}$$

Lemma 1 (Equilibria). $\psi = -\frac{i}{2} c_\gamma \log \rho$

$$\begin{cases} \mathrm{d}Z_t = i\sigma^2 \nabla_{\bar{\mathbf{z}}} \psi(Z_t, \bar{Z}_t)\,\mathrm{d}t + \sigma\,\mathrm{d}W_t \\ \mathrm{d}\bar{Z}_s = i\sigma^2 \nabla_{\mathbf{z}} \psi(Z_s, \bar{Z}_s)\,\mathrm{d}s + \sigma\,\mathrm{d}\bar{W}_s \end{cases}$$





https://circle-group.github.io/research/CSB/

Conformal Optimal Transport

$$\mathrm{COT}_\gamma(\rho_0,\rho_T): \inf_{\pi\in\Pi(\rho_0,\rho_T)} \iint_{\mathbb{C}^d\times\mathbb{C}^d} \mathcal{C}_\gamma(\mathbf{z}_0,\mathbf{z}_T)\,\mathrm{d}\pi(\mathbf{z}_0,\mathbf{z}_T) + \sigma^2\mathrm{KL}(\pi\|\rho_0\otimes\rho_T),$$

$$\mathcal{C}_\gamma(\mathbf{z}_0,\mathbf{z}_T) := \inf_{\substack{u(0)=\mathbf{z}_0\\ u(T)=\mathbf{z}_T}} \int_0^T L_\gamma(u(\tau),\dot{u}(\tau),\tau)\,\mathrm{d}\tau, \quad L_\gamma(\mathbf{z},\mathbf{p},t;\mathbf{z}_T) = \left\|\mathbf{p}-b_\gamma(\mathbf{z},t;\mathbf{z}_T)\right\|^2$$

Girsanov Theorem ($\mathbb{P}: \mathrm{d}Z_t = i\mathsf{A}_t\,\mathrm{d}t + \sigma\,\mathrm{d}W_t$).

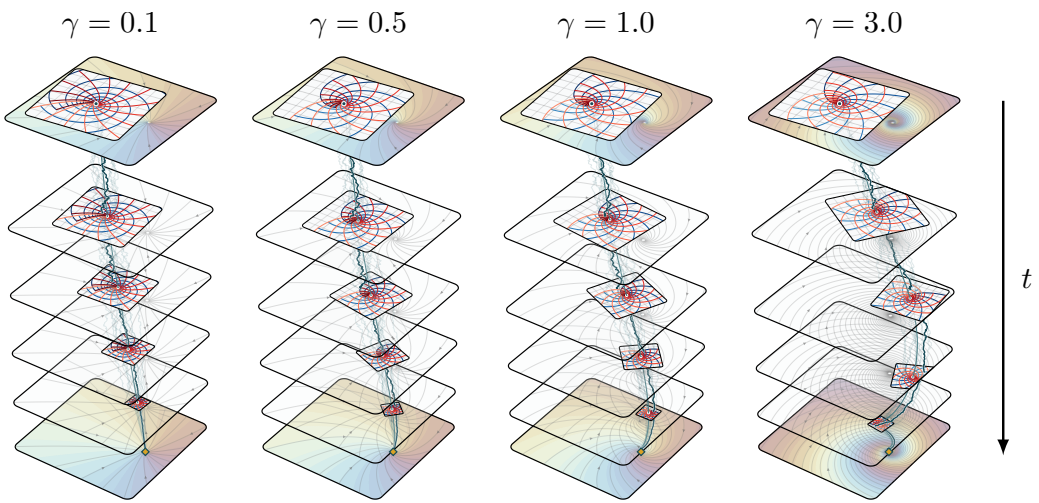
$$\log\frac{\mathrm{d}\mathbb{P}}{\mathrm{d}\mathbb{Q}_W}\bigg|_{\mathcal{F}_t} = 2\Re\left[\int_0^t(\bar{\vartheta}_\tau)^\mathsf{T}\,\mathrm{d}W_\tau\right] - \int_0^t\|\vartheta_\tau\|^2\,\mathrm{d}\tau, \qquad \vartheta_t = \frac{i\mathsf{A}_t}{\sigma}$$

KL Objective

$$\underset{\mathbb{P}^\gamma_{0,T}=\rho_{0,T}}{\text{minimize}}\,\mathrm{KL}(\mathbb{P}^\gamma\|\mathbb{Q}^\gamma_W),$$

$$\text{s.t.}\quad \mathbb{Q}^\gamma_W = (\mathcal{H}_\gamma)_\#(\mathbb{Q}_W).$$

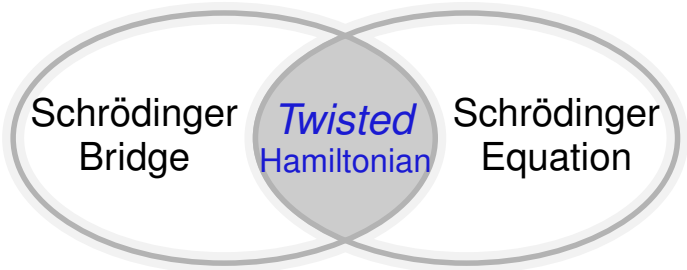
- $\mathbb{Q}_W$: a reference measure.
- $\mathcal{H}_\gamma$: a quasiconformal mapping.



COT to CSB

Twisted Hamiltonian: $H(\cdot, \cdot, \nabla_{\mathbf{z}}\psi, \nabla_{\bar{\mathbf{z}}}\psi, \rho)$

$$\sigma^2 \langle \nabla_{\mathbf{z}}\psi, \nabla_{\mathbf{z}}\bar{\psi} \rangle + \nabla_{\mathbf{z}}\psi^{\mathsf{T}} f - \nabla_{\bar{\mathbf{z}}}\psi^{\mathsf{T}} \bar{f} + \frac{i\gamma\sigma^2}{2} \frac{\Delta\sqrt{\rho}}{\sqrt{\rho}}$$



Conformal OT

$$\underset{\mathbb{P}^{\gamma}_{0,T}=\rho_{0,T}}{\text{minimize}}\ \mathrm{KL}(\mathbb{P}^{\gamma} \| \mathbb{Q}^{\gamma}_{W})$$

Complex Hopf–Cole

Hamilton–Jacobi

$$i\partial_t\psi = H(\mathbf{z}, \bar{\mathbf{z}}, \nabla_{\mathbf{z}}\psi, \nabla_{\bar{\mathbf{z}}}\psi, \rho) - \frac{i}{4} c_\gamma \Delta(\sigma^2\psi)$$

Fokker–Planck

$$\partial_t\rho = i\nabla^{\dagger}_{\bar{\mathbf{z}}}\{(\nabla_\alpha H)\,\rho\} - i\nabla^{\dagger}_{\mathbf{z}}\{(\overline{\nabla_\alpha H})\,\rho\} + \frac{1}{4}\Delta(\sigma^2\rho)$$

∇_α: gradient w.r.t. $\alpha = \nabla_{\mathbf{z}}\psi$

Let $\Phi = \exp(i\psi/c_\gamma)$ and $\hat{\Phi} = \rho/\Phi$ such that $\Phi_0\hat{\Phi}_0 = \rho_0$ and $\Phi_T\hat{\Phi}_T = \rho_T$.

Proposition 1 (CSB-PDEs).

$$\begin{cases} \frac{\partial\Phi(\mathbf{z},\bar{\mathbf{z}},t)}{\partial t} = -i\nabla_{\mathbf{z}}\Phi^{\mathsf{T}} f + i\nabla_{\bar{\mathbf{z}}}\Phi^{\mathsf{T}}\bar{f} - \frac{\sigma^2}{4}c_\gamma\Delta\Phi + \frac{i\gamma\sigma^2}{2c_\gamma}\frac{\Delta\sqrt{\rho}}{\sqrt{\rho}}\Phi, \\ \frac{\partial\hat{\Phi}(\mathbf{z},\bar{\mathbf{z}},t)}{\partial t} = -i\nabla_{\mathbf{z}}\cdot(\hat{\Phi}f) + i\nabla_{\bar{\mathbf{z}}}\cdot(\hat{\Phi}\bar{f}) + \frac{\sigma^2}{4}c_\gamma\Delta\hat{\Phi} - \frac{i\gamma\sigma^2}{2c_\gamma}\frac{\Delta\sqrt{\rho}}{\sqrt{\rho}}\hat{\Phi}, \end{cases}$$

Complex Schrödinger Bridges

Find nonlinear drift such that $Z_0 \sim \rho_0$, $Z_T \sim \rho_T$

$$\begin{aligned} \mathrm{d}Z_t &= i\big(f(Z_t,t) + \sigma^2 \nabla_{\bar{\mathbf{z}}} \psi(Z_t, \bar{Z}_t, t)\big)\,\mathrm{d}t + \sigma\,\mathrm{d}W_t \\ \mathrm{d}\bar{Z}_s &= i\big(\bar{f}(Z_s, T\text{-}s) + \sigma^2 \nabla_{\mathbf{z}} \psi(Z_s, \bar{Z}_s, T\text{-}s)\big)\,\mathrm{d}s + \sigma\,\mathrm{d}\bar{W}_s \end{aligned}$$

Complex Hopf–Cole ansatz

$$\begin{aligned} &\psi = \chi + \gamma \log \rho, \quad \chi + \eta = -i \log \rho, \\ &(v_t, \hat{v}_t, \check{v}_t, \bar{v}_t) = \sigma(\nabla_{\mathbf{z}} \chi_t, \nabla_{\bar{\mathbf{z}}} \chi_t, \nabla_{\bar{\mathbf{z}}} \eta_t, \nabla_{\mathbf{z}} \eta_t). \end{aligned}$$

Theorem 1 (General SB-CVSDEs).

$$\begin{cases} \mathrm{d}Z_t = i\big(f_t + \sigma^2 \nabla_{\bar{\mathbf{z}}} \psi_t(Z_t, \bar{Z}_t, t)\big)\,\mathrm{d}t + \sigma\,\mathrm{d}W_t, \\ \mathrm{d}Y_t = -i\Big(\|v_t\|^2 + F_t + \gamma\sigma \nabla_{\mathbf{z}} \cdot \check{v}_t - \frac{i\gamma}{2}\|\hat{v}_t + \check{v}_t\|^2\Big)\,\mathrm{d}t + v_t^{\mathsf{T}}\,\mathrm{d}W_t + \hat{v}_t^{\mathsf{T}}\,\mathrm{d}\bar{W}_t, \\ \mathrm{d}\widehat{Y}_t = -i\Big(\|v_t\|^2 + F_t - \gamma\sigma \nabla_{\mathbf{z}} \cdot \hat{v}_t - (1 + \frac{i\gamma}{2})\|\hat{v}_t + \check{v}_t\|^2\Big)\,\mathrm{d}t - \\ \qquad\qquad \Big(\sigma \nabla_{\mathbf{z}} \cdot \check{v}_t + \sigma \nabla_{\bar{\mathbf{z}}} \cdot \bar{v}_t\Big)\,\mathrm{d}t - \bar{v}_t^{\mathsf{T}}\,\mathrm{d}W_t - \check{v}_t^{\mathsf{T}}\,\mathrm{d}\bar{W}_t, \qquad Y_t = \chi_t, \quad \widehat{Y}_t = -\eta_t. \end{cases}$$

➡ **General CSB algorithm**

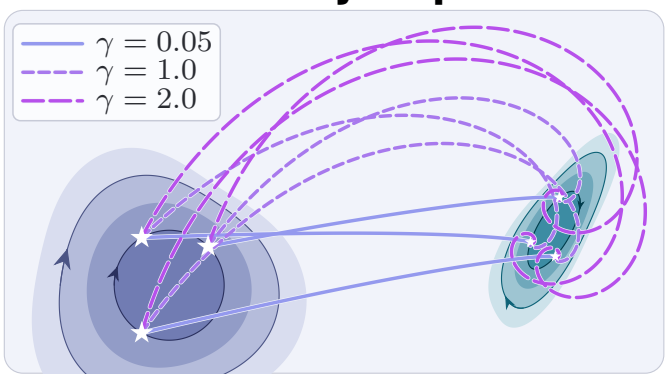
Let $\hat{\omega}_s^2 = \int_0^s \hat{\sigma}_\tau^2\,\mathrm{d}\tau$, $\omega_s^2 = \int_s^T \hat{\sigma}_\tau^2\,\mathrm{d}\tau$, and $\bar{c}_\gamma = 1 - i\gamma$.

Proposition 2 (Analytic posterior).

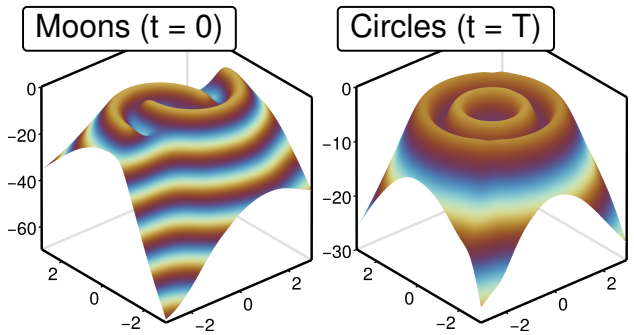
$$\begin{aligned} q(\bar{\mathbf{z}}_s | \bar{\mathbf{z}}_T, \bar{\mathbf{z}}_0) &= \mathcal{CN}(\bar{\mathbf{z}}_s | \bar{\mu}_s, \Sigma_s), \\ \bar{\mu}_s = \bar{\mathbf{z}}_T + (\bar{\mathbf{z}}_0 - \bar{\mathbf{z}}_T)&\left(\frac{\hat{\omega}_s^2}{\hat{\omega}_s^2 + \omega_s^2}\right)^{\bar{c}_\gamma}, \quad \Sigma_s = \left(\frac{\omega_s^2 \hat{\omega}_s^2}{\omega_s^2 + \hat{\omega}_s^2}\right)\mathbf{I}_d. \end{aligned}$$

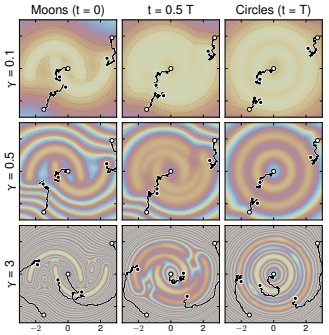
➡ **Conditional CSB algorithm**

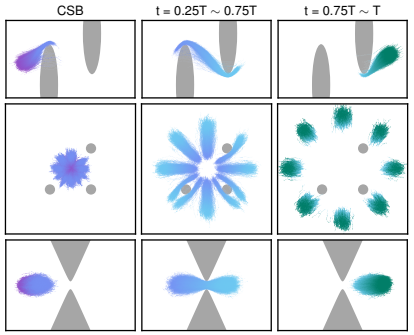
Pairwise analytic posterior



Experimental Results

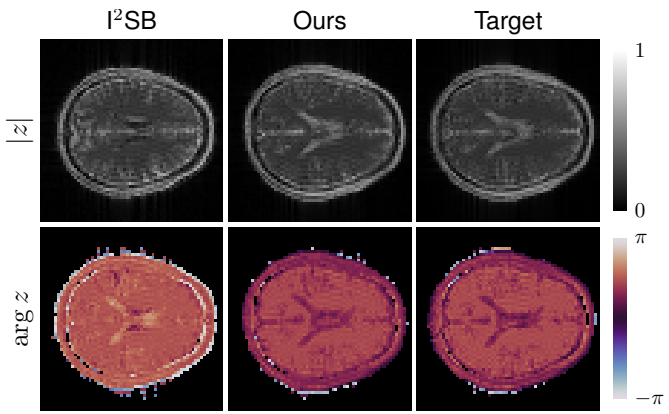


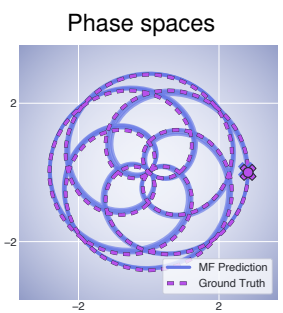


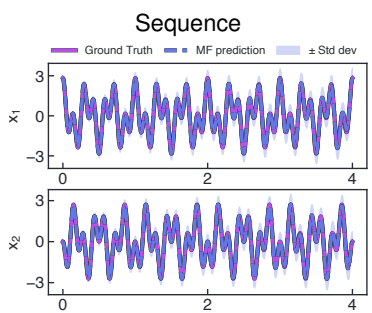


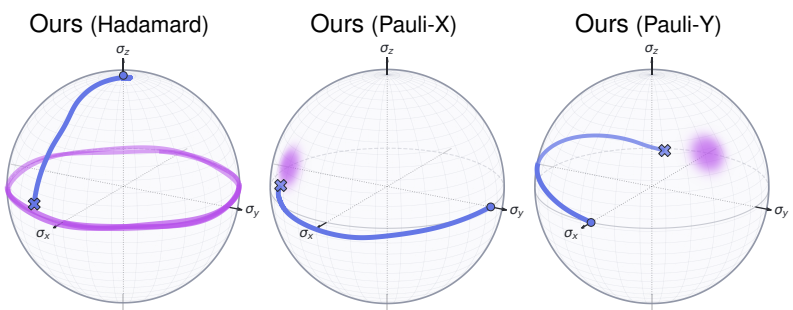




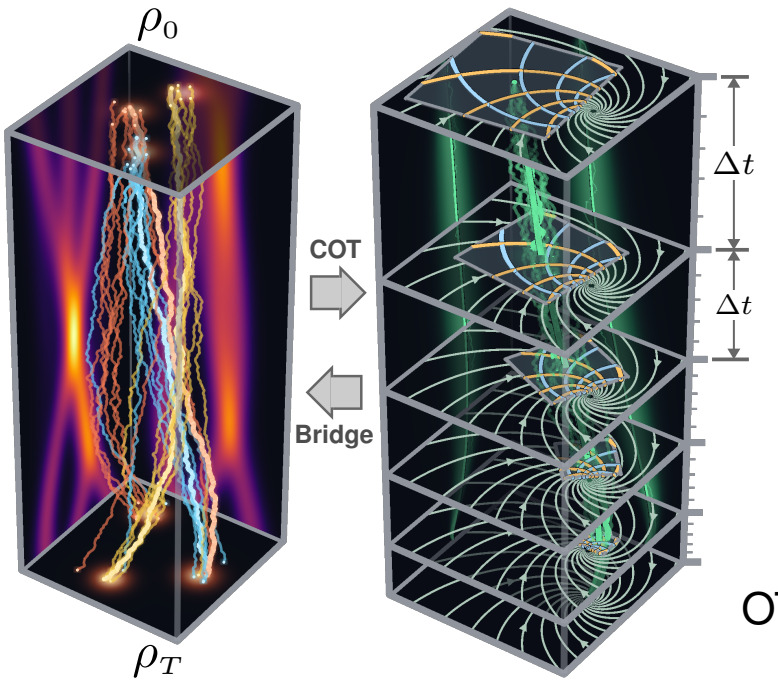


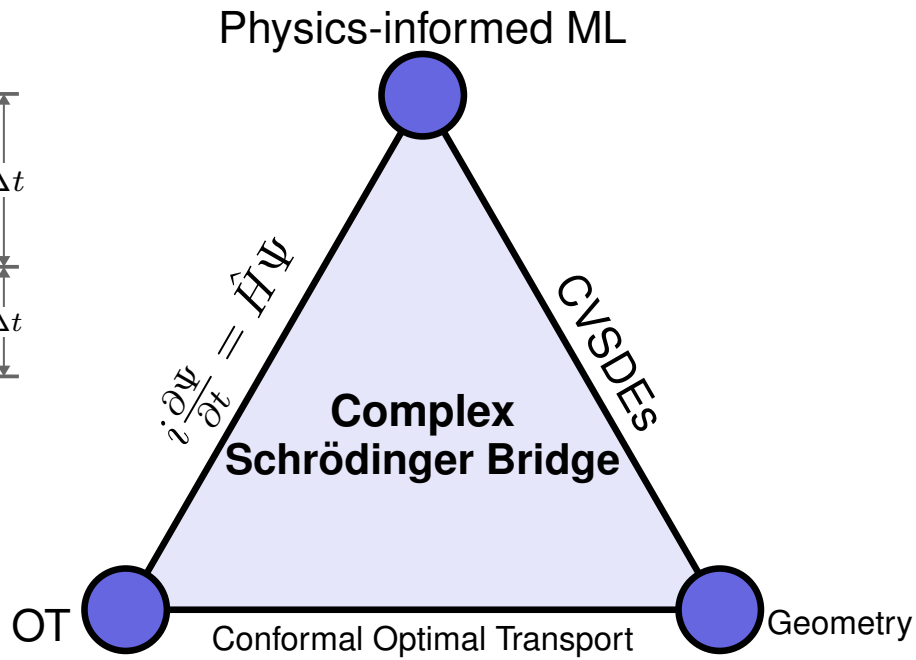






Complex Schrödinger Bridges





`https://circle-group.github.io/research/CSB`

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