Free Decompression with Algebraic Spectral Curves

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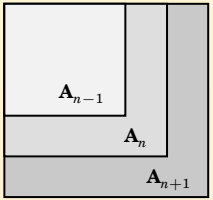
NeurIPS 2026

Motivation: Spectral densities, edges, and atoms reveal important properties of ML models.

Challenges:

- **Direct eigendecomposition** costs $\mathcal{O}(n^3)$; expensive for large matrices.
- **Matrix formation** can also hit the $\mathcal{O}(n^2)$ **memory wall**.

Concept



Suppose matrix of interest is embedded in an infinite sequence of nested matrices

$$\mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_3, \ldots \quad \text{with } \mathbf{A}_n \in \mathbb{R}^{n \times n} \quad \text{such that} \quad (\mathbf{A}_n)_{ij} = (\mathbf{A}_{n+1})_{ij}.$$

Objective: Infer the spectral distribution of a **large matrix** $\mathbf{A}_n$ from a much smaller **principal submatrix** $\mathbf{A}_{n_0}$, where $n_0 \ll n$.

Free Decompression (Previous Work)

Spectral Representation:

- We encode the spectrum of $\mathbf{A}_n$ by its Stieltjes transform m_n.
- In the large-matrix limit, $m_n \to m$, which determines the spectral density ρ.

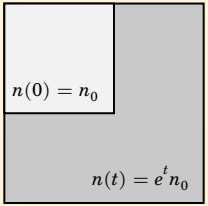
Stieltjes Transform

$$m_n(z) = \frac{1}{n}\mathsf{trace}(\mathbf{A}_n - z\mathbf{I})^{-1}$$

Inverse Stieltjes Transform

$$\rho(x) = \frac{1}{\pi}\lim_{\varepsilon \to 0^+} \Im m(x + i\varepsilon)$$

Free Decompression



- Consider a family of matrices $\mathbf{A}_{n(t)}$ with size $n(t) = e^t n_0$.
- Then the transform $m(t, z)$ satisfies

$$\frac{\partial m}{\partial t} = -m + \frac{1}{m}\frac{\partial m}{\partial z}$$

Free decompression (Ameli et al., NeurIPS 2025):

1. **Initial condition:** estimate $m(0, \cdot)$ from the submatrix $\mathbf{A}_{n_0}$;
2. **Solve PDE:** evolve $m(t, \cdot)$ to $T = \log(n/n_0)$;
3. **Inverse transform:** recover the spectral density of $\mathbf{A}_n$ from $m(T, \cdot)$.

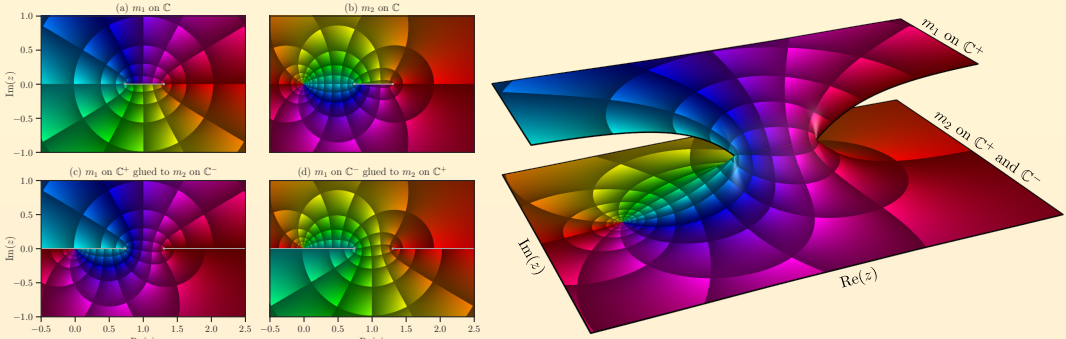
Algebraic Spectral Curves

- ***Challenge:*** Solving the FD PDE requires analytic continuation across branch cuts, which is **numerically ill-posed**.
- ***Limitation:*** Previous FD was restricted to simple, single-bulk spectra.
- ***Our solution:*** Restrict the Stieltjes transform to **algebraic functions**, satisfying a polynomial relation

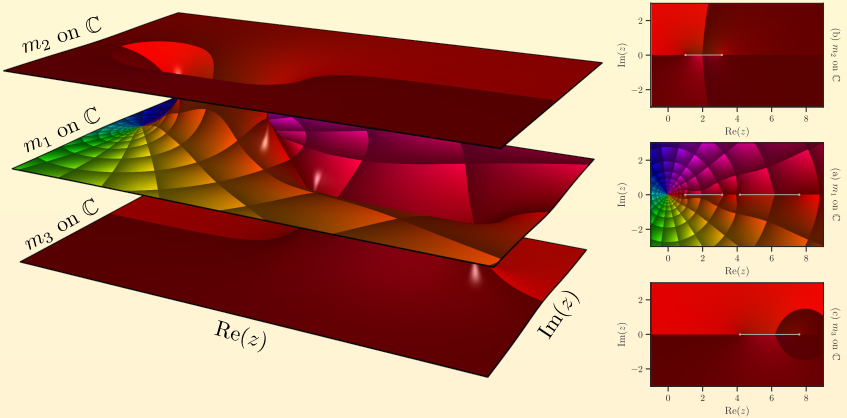
$$P(z, m) = 0.$$

- ***Geometric interpretation:*** The branches of P form multiple sheets, glued along branch cuts into a **spectral curve**:

$$\mathcal{C} = \left\{ (z, m) \in \mathbb{C}^2 \mid P(z, m) = 0 \right\}$$



Example of an Algebraic Spectral Curve



- **Compound free Poisson** law with two bulks, represented by a **cubic** polynomial.
- *Right:* physical **Stieltjes** branch m_1 (center), non-physical branches m_2, m_3 (top/bottom).
- *Left:* **spectral curve** (gluing sheets at branch cuts), a complex curve in $\mathbb{C}^2 \cong \mathbb{R}^4$.

Free Decompression on Spectral Curves

Theorem (Algebraic Free Decompression)

If the initial Stieltjes transform satisfies $P(z, m) = 0$, then after decompression by $\tau = n/n_0$, the evolved transform m_τ satisfies

$$P\left(z + \frac{\tau - 1}{\tau m_\tau(z)},\ \tau m_\tau(z)\right) = 0.$$

Advantages

- Algebraic structure is **closed** under FD.
- Multi-bulk, multi-modal, multi-scale, and atomic spectra.

Additional Numerical Contributions

- **Stable polynomial fitting** from empirical spectra.
- **Physical-branch tracking** via tangent continuation.

Beyond the Density

Direct Evolution of Spectral Quantities: without reconstructing the full density. Define

$$F_1 = P(\zeta, y), \qquad F_2 = y^2 \partial_y P - (\tau - 1)\partial_\zeta P.$$

Spectral Edges

$$F_1 = F_2 = 0$$

Track spectral bulk boundaries.

Cusps & Bulk Splitting

$$F_1 = F_2 = 0$$
$$\det \partial(F_1, F_2)/\partial(\zeta, y) = 0$$

Identify splitting time/location.

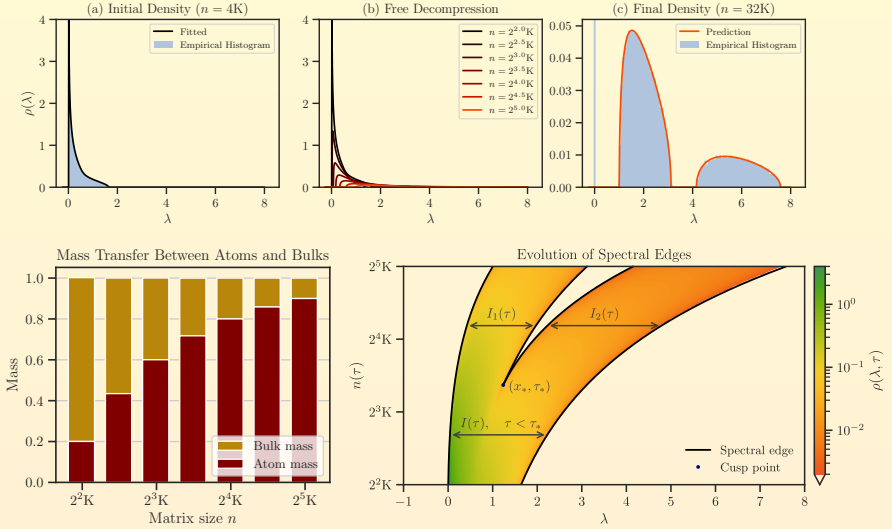
Spectral Atoms

$$x_\circ(\tau) = x_\circ$$
$$w(\tau) = 1 - \tau^{-1}(1 - w_\circ)$$

Exact evolution of atomic mass.

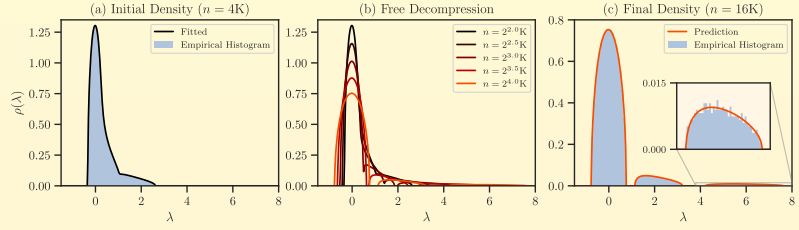
Spectral Moments: Direct evolution from initial moments, without reconstructing the density.

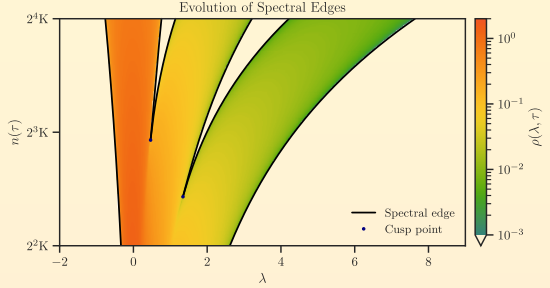
Examples: Compound Free Poisson Law



Density evolution (top), **atom/bulk mass** (bottom left), and **spectral edges** (bottom right)

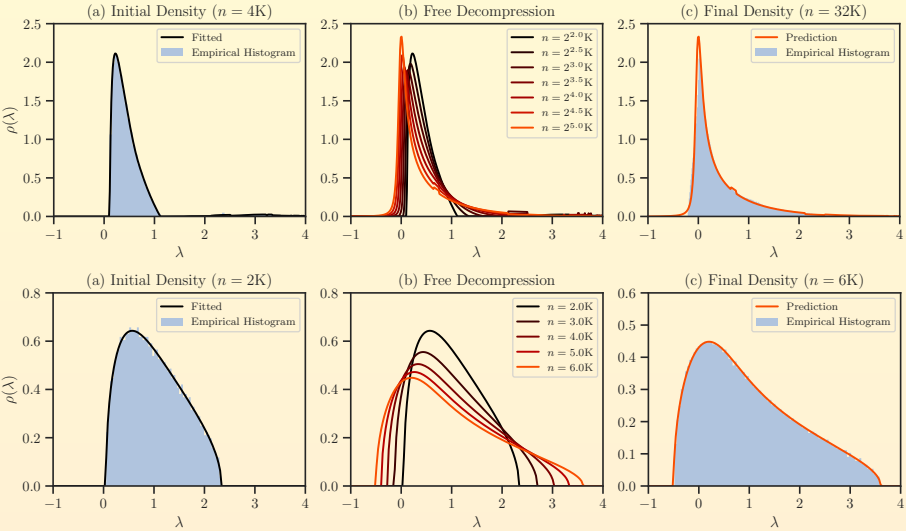
Examples: Free Lévy Law





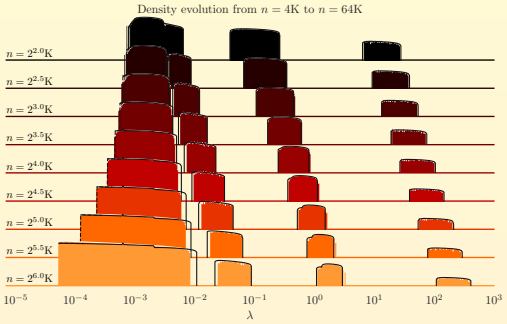
Density evolution (top) and **spectral edges** (bottom)

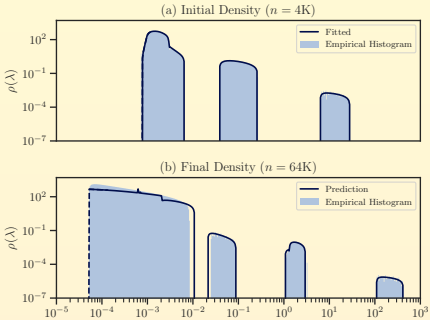
Examples: Hessian

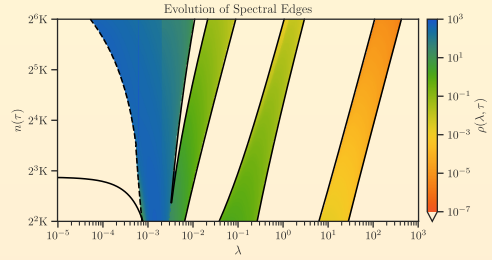


Autoencoder Hessian (top) and **Pennington–Bahri Hessian model** (bottom)

Examples: Diffusion Model







- Random-feature **diffusion**: expected activation matrix
- FD expands $n_0 = 4000$ to $n = 64{,}000$
- Atom-like peak treated as a *continuous bulk*, not an atom
- **Edges** relate to *generalization*/*memorization* timescales (Bonnaire et al., NeurIPS 2025)

Size	Process Time (sec)		Distrib. Dist.		Moments Error		Edges Error	
n	Direct	FD (ours)	W_1/L	MMD	$\Delta\mu_1/\mu_1$	$\Delta\sigma/\sigma$	Mean	Max
2^2K	32.2	$\mathbf{32.2 + 0.0}$	0.01%	0.01%	0.01%	0.01%	0.1%	0.2%
2^3K	431.43	$\mathbf{32.2 + 2.3}$	0.22%	0.08%	0.02%	0.03%	0.2%	0.5%
2^4K	3534.5	$\mathbf{32.2 + 2.9}$	0.43%	0.12%	0.01%	0.00%	0.2%	0.2%
2^5K	27056.2	$\mathbf{32.2 + 4.3}$	0.70%	0.15%	0.05%	0.08%	0.4%	0.8%
2^6K	192447.7	$\mathbf{32.2 + 6.8}$	2.19%	0.53%	0.08%	0.09%	0.4%	1.2%

- Showing runtime and accuracy results for **Diffusion model** (previous slide)
- Runtime with direct method increases by $\mathcal{O}(n^3)$.
- Runtime with **free decompression (FD)** is an initial overhead $\mathcal{O}(n_0^3)$ plus $\mathcal{O}(1)$.

Resources

This Work

Free Decompression with Algebraic Spectral Curves
Ameli, S., van der Heide, C., Hodgkinson, L., Mahoney, M.W. *NeurIPS 2026*

Related Works

Spectral Estimation with Deformed Decompression
Ameli, S., van der Heide, C., Hodgkinson, L., Mahoney, M.W. *NeurIPS 2026*

Spectral Estimation with Free Decompression
Ameli, S., van der Heide, C., Hodgkinson, L., Mahoney, M.W. *NeurIPS 2025*

Software

Package	Documentation	Install
freealg	`ameli.github.io/freealg`	`pip install freealg`